Model Reduction for Burgers Equation

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May 28, 2019

Overview

Motivation

- Consider a class of parametried partial differential equations (P^2DEs)

$$\partial_t u(t;\mu) + \mathcal{F}[u(t;\mu);\mu] = 0 \tag{1}$$

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- Question: how to find X_k? Reduced Basis Method.

Complexity Reduction

Definition

Let X be a normed linear space, S be a subset of X and X_n be a generic n-dimensional subspace of X . The deviation of S from X_n is

$$E(S; X_n) = \sup_{u \in S} \inf_{v_n \in X_n} \|u - v_n\|_X \tag{3}$$

The Kolmogorov n-width of S in X is given by

$$d_n(S, X) = \inf_{X_n} \sup_{u \in S} \inf_{v_n \in X_n} \|u - v_n\|_X \tag{4}$$

Complexity Reduction

Remark

- *The n-width of S thus measures the extent to which S may be approximated by a n-dimensional subspace of X.*
- $S = \{u(.;\mu); \mu \in \mathcal{P}_{train}\}$ *called solution manifold.*
- *We further assume that S has a small Kolmogorov n-width* [a]*.*
- *PDEs hyperbolic always have a large Kolmogorov n-width.*

[a] If $\mathcal{F}$ is a bounded linear operator mapping the Banach space X into the Banach space Y and D is a compact set in X , then the Kolmogorov widths of the image $L(D)$ do not exceed those of D multiplied by the norm of L.

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The homogeneous advection equation

$$\begin{cases} \partial_t u + c\partial_x u = 0; \\ u\big|_{t=0} = u_0; \end{cases} \tag{5}$$

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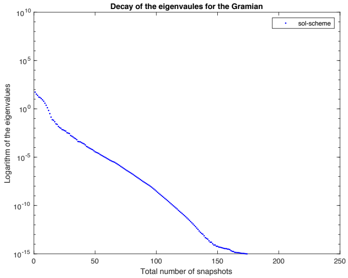

(a) Typical Kolmogorov n-width for PDE hyperbolic

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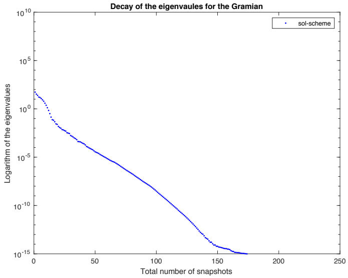

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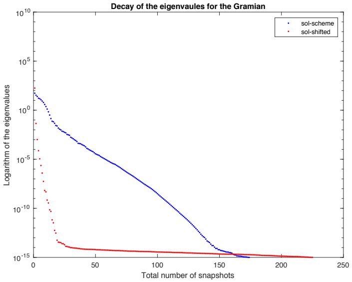

(b) A sequence of small dimensional subspaces(in red)

Overview

Inviscid Burgers 1D

Let $(x, t) \in \Omega \times [0, T]$, we focus on the Cauchy problem:

$$\begin{cases} \partial_t u + u \partial_x u = 0 \\ u(x, 0; \mu) = u_0(x; \mu) \end{cases} \tag{6}$$

- $\mu \in \mathcal{P}_{\text{train}} \subset \mathbb{R}^d$
- $u_0(\cdot, \mu) \in \mathcal{L}^\infty(\Omega)$
- Non-linear hyperbolic PDE
- The initial data u^0 to be parameter-separable

$$u_0(x; \mu) = \sum_{q=1}^{Q_{u_0}} \theta_q(\mu) f_q(x)$$

Method of characteristics

Definition

The characteristic for the dynamic of u is an absolutely continuous function $\tau \mapsto X(\tau; x, t)$ which satisfies $X(t; x, t) = x$ and the ordinary differential equation

$$\frac{\mathrm{d}X(\tau; x, t)}{\mathrm{d}\tau} = u(X(\tau; x, t), \tau) \tag{7}$$

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To be solved:

$$x - X(0; x, t) = u_0(X(0; x, t))t \tag{9}$$

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- *To ensure the well-posedness of the problem, we need* $T \left\| u_0'(\cdot, \mu) \right\|_{L^\infty(\Omega)} < 1$;
- $u_0(\cdot, \mu)$ *may very well be non-linear function, which makes the complexity of the computation is still very high.*

Empirical Interpolation Method[1]

Step $n = 1$:

$$\text{Find} \begin{cases} v_1 = \arg\max_{v \in S} \|v\| \\ x_1 = \arg\max_{x \in \Omega} |v_1(x)| \end{cases}$$

$$\text{and set} \begin{cases} q_1 := \dfrac{v_1}{v_1(x_1)} \quad \text{(normalization)} \\ P_1 := \{x_1\} \\ \mathbb{X}_1 := \text{span}\{v_1\} = \text{span}\{q_1\} \end{cases}$$

[1] Y.Maday et al. An 'empirical interpolation' method: application to efficient reduced-basis discretization of partial differential equations. Comptes Rendus Mathematique. Volume 339, Issue 9, 1 November 2004, Pages 667-672

Empirical Interpolation Method[2]

Step $n > 1$:

$$\text{Find } \begin{cases} v_n = \arg\max_{v \in S} \|v - \mathcal{I}_{n-1}[v]\| \\ x_n = \arg\max_{x \in \Omega} |v_n(x)| \end{cases}$$

$$\text{and set } \begin{cases} q_n := \dfrac{v_n - \mathcal{I}_{n-1}[v_n]}{v_n(x_n) - \mathcal{I}_{n-1}[v_n](x_n)} \quad \text{(normalization)} \\ P_n := P_{n-1} \cup \{x_1\} \\ \mathbb{X}_n := \text{span}\{\mathbb{X}_{n-1}, q_n\} \end{cases}$$

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Empirical Interpolation Method

In:

- A large set of snapshots $X^0_{\text{train}} := \{X_0(\cdot, \mu_i)\}_{i=1}^{N_{\text{train}}}$, which is precomputed for a training set of parameter μ_i.

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Out:

- a set of interpolation points $P_M := \{x_m\}_{m=1}^{M} \subset \Omega$
- a corresponding nodel interpolation basis $Q_M := \{q_m\}_{m=1}^{M}$ i.e. $q_m(x_{m'}) = \delta_{m,m'}$, $1 \leq m, m' \leq M$.

Numerical Test

$\Omega = [0, 1]$ and $T = 1$, the grid decomposition for $\Omega \times [0, T]$ is 100×100, μ random in $[0, 1]$ and $N_{\text{train}} = 120$.

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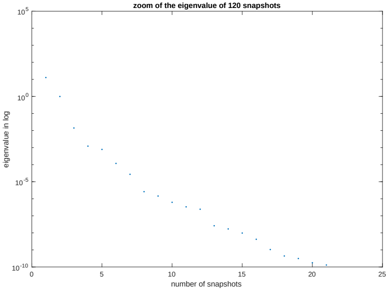

(a) Eigenvalues of POD or SVD

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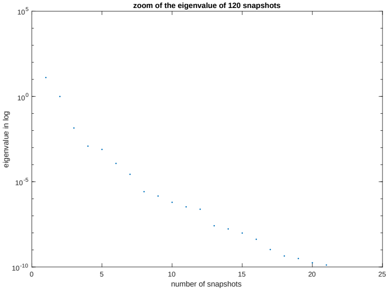

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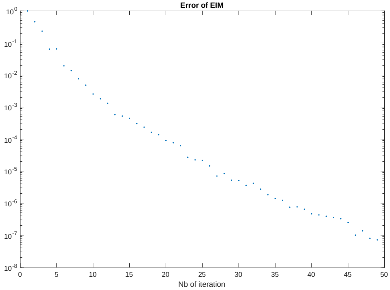

(b) Error of the EIM algorithm

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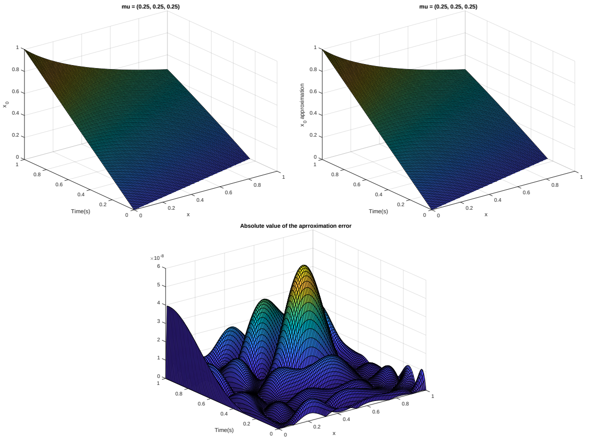

Figure: The exact solution with the approximate solution for x_0 and absolute error.

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Viscid Burgers 2D

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{10}$$

- add some diffusive effects in the current model.

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- no longer a hyperbolic PDE but a parabolic PDE.
- the method of characteristics is no longer available.

Scheme

Inspiration [3]

$$\frac{\partial u}{\partial t} + u \cdot \nabla u \to \frac{\mathrm{d}u}{\mathrm{d}t}$$

[3] O.Pironneau. On the Transport Diffusion Algorithm and Its Applications to the Navier-Stokes Equations. Numer. Math. 38, 309-332 (1982)

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Splitting scheme:

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = u(X,t) & t \in]t_n, t_{n+1}[\\ (u_{n+1}, v) + \mu\Delta t(\nabla u_{n+1}, \nabla v) = (u_n(X_n(\cdot)), v) & v \in \mathcal{H}_0^1(\Omega) \end{cases} \quad (11)$$

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Full order problem:

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Reduced order problem:

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Numerical Test

Numerical settings[4]:
$\Omega = [0,2] \times [0,1]$, $T = 0.3$, the grid decomposition is 120×60 and $dt = 0.01$. Initial condition: $u_0(x,y) = \frac{1}{2}(1 + \sin(2\pi x)\sin(2\pi y))$
Simulation:

Click for video

[4] B. Haasdonk and M. Ohlberger. Reduced Basis Method for Explicit Finite Volume Approximations of Nonlinear Conservation Laws. Proceedings of Symposia in Applied Mathematics. 2008

Overview

Example

Heat equation:

$$\begin{cases} \partial_t u - \mu \Delta u = 0 & \text{in } \Omega \times (0, T) \\ u(0) = u_0 & \text{in } \Omega \end{cases} \tag{14}$$

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Residual:

$$\mu \Delta t R_n(v) := (u^r_{n+1}, v) + \mu \Delta t (\nabla u^r_{n+1}, \nabla v) - (u^r_n, v), \quad v \in \mathcal{W} \tag{17}$$

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Cauchy-Schwartz and Young's inequality for (e_n, e_{n+1}), we find

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Finally, sommation over n

$$\|e_N\|^2 + \mu\Delta t \sum_{n=1}^{N} \|\nabla e_n\|^2 \leq \|e_0\|^2 + \mu C\Delta t \sum_{n=0}^{N-1} \|R_n\|^2 \tag{20}$$

POD-Greedy[5]

Let $\varepsilon_{\text{tol}} > 0$ a given error tolerance, set $X_0 := \{0\}$, $\Phi_0 := \emptyset$

while $\varepsilon_n := \max_{\mu \in \mathcal{P}_{train}} \Delta(X_n, \mu) > \varepsilon_{tol}$ **do**

$\mu^{n+1} := \arg\max_{\mu \in \mathcal{P}_{\text{train}}} \Delta(X_n, \mu)$

$u_{n+1}^k := u^k(\mu^{(n+1)})$, $k = 0, \cdots, K$ solution of the full problem

$e_{n+1}^k := u_{n+1}^k - P_{X_n} u_{n+1}^k$, $k = 0, \cdots, K$

$\phi_{n+1} := \text{POD}_1(\{e_{n+1}^k\}_{k=0}^K)$

$\Phi_{n+1} := \Phi_n \cup \{\phi_{n+1}\}$

$X_{n+1} := X_n + \text{span}(\phi_{n+1})$

end

[5] B.Haasdonk. and M.Ohlberger. Reduced Basis Method for Finite Volume approximations of parametrized linear evolution equations. ESAIM: M2AN 42 (2008) 277–302

Overview

Conclusion

What we have done:

- RB-method with method of characteristics
- Splitting scheme
- RB-method on our scheme

What we need to do:

- Estimation a-posteriori for this problem
- What happens when we have shocks