Model Reduction for hyperbolic Equations

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Overview

Motivation

- Considering a class of parametried partial differential equations (P^2DEs)

$$\partial_t u(t;\mu) + \mathscr{F}[u(t;\mu);\mu] = 0 \tag{1}$$

where $\mu \in \mathbf{R}^d$ a parameter vector.

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- Question: how to find X_k? Reduced Basis Method.

Complexity Reduction

Definition

Let X be a normed linear space, S be a subset of X and X_n be a generic n-dimensional subspace of X . The deviation of S from X_n is

$$E(S;X_n) = \sup_{u \in S} \inf_{v_n \in X_n} \|u - v_n\|_X$$

The Kolmogorov n-width of S in X is given by

$$d_n(S,X) = \inf_{X_n} \sup_{u \in S} \inf_{v_n \in X_n} \|u - v_n\|_X \tag{2}$$

Complexity Reduction

Remark

- *The n-width of S thus measures the extent to which S may be approximated by a n-dimensional subspace of X.*
- $S = \{u(.;\mu); \mu \in \mathscr{P}_{train}\}$ *called solution manifold.*
- *We further assume that S has a small Kolmogorov n-width* [1]*.*
- *PDEs hyperbolic always have a large Kolmogorov n-width.*

[1] If $\mathscr{F}$ is a bounded linear operator mapping the Banach space X into the Banach space Y and D is a compact set in X , then the Kolmogorov widths of the image $L(D)$ do not exceed those of D multiplied by the norm of L.

Exemple

The homogeneous advection equation

$$\begin{cases} \partial_t u + c\partial_x u = 0; \\ u\big|_{t=0} = u_0; \\ u \quad periodic \end{cases}$$

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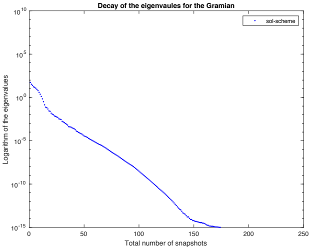

(a) Typical Kolmogorov n-width for PDE hyperbolic

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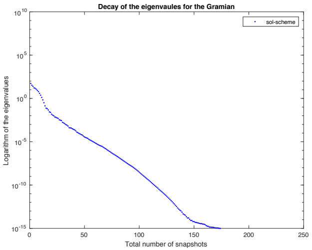

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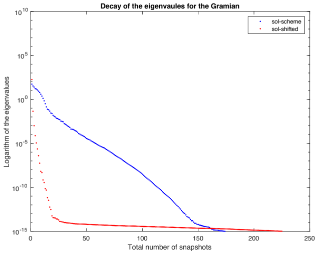

(b) A sequence of small dimensional subspaces(in red)

Overview

Primary Problem

$$\begin{cases} \frac{d}{dt}u(t) = Au(t) + F(u(t)), \\ u(0) = u_0, \end{cases} \tag{3}$$

- $u(t) = u(t;\mu) \in \mathbf{R}^N$: a state vector;
- $u_0 \in \mathbf{R}^N$: a fixed initial condition;
- $A \in \mathbf{R}^{N\times N}$: a square matrix;
- $F : \mathbf{R}^N \mapsto \mathbf{R}^N$: a non-linear function,
- $\mu \in \mathscr{P}_{train}$, a parameter.

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Best solution: POD!

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- Define the Gramian matrix:

$$G_{ij} := \int_0^T u(t;\mu_i)^T u(t;\mu_j) dt, \text{ with } \mu_i, \mu_j \in \mathscr{P}_{train}.$$

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- Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N_\mu} \geq 0$ denote the ordered eigenvalues of G and $\phi_i \in \mathbf{R}^{N_\mu}$, $i = 1, \cdots, N_\mu$ denote their associated eigenvectors which are also referred to as the POD modes
$$G\phi_i = \lambda_i \phi_i$$

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- The POD modes $\Psi := S\Phi\Lambda^{-\frac{1}{2}}$;
- We then define our projector Π_k by taking the first k POD basis of Ψ with a smaller dimension k.

Reduced Problem

With this projector Π_k, we can project the solution onto the this subspace $X_k = span\{\psi_1, \cdots, \psi_k\}$

$$u(t) = \underbrace{\Psi_k}_{N\times k} \underbrace{\tilde{u}(t)}_{k\times 1}$$

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Let $f(t) := F(\Psi_k \tilde{u}(t))$ the non-affine parameter dependent part, we would like to find an approximation

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How to choose a m by m linear system: DEIM!

Algorithm 1 DEIM

Input: the projection basis U

Output: the interpolation indices $\vec{\varrho}$

1: $[\rho, \varrho_1] = max\{|u_1|\}$;
2: $\vec{\varrho} := [\varrho_1], \mathbf{U} := [u_1]$;
3: **for** $i = 2, \cdots, m$ **do**
4: $\quad u = u_i$;
5: $\quad \mathbf{U}_{\vec{\varrho}} c = u_{\vec{\varrho}}$;
6: $\quad r := u - \mathbf{U} c$;
7: $\quad [\rho, \varrho_i] = max\{|r|\}$;
8: $\quad \mathbf{U} := [\mathbf{U}, u],\ \vec{\varrho} := [\vec{\varrho}, \varrho_i]$
9: **end for**

Reduced problem

Let $P = [e_{\varrho_1}, \cdots, e_{\varrho_m}]$ where e_i is the standard basis of $\mathbf{R}^N$.

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Now the precomputation can be done and the complexity of the non-linear term

$$\Psi_k \in \mathbf{R}^{N\times k} \Rightarrow P^T \Psi_k \in \mathbf{R}^{m\times k}$$

Overview

Simple Model

Homogenous Burgers' Equation:

$$\begin{cases} \partial_t u + u\partial_x u = 0 \\ u(x,0) = u^0(x) \\ u \quad \text{periodic} \end{cases} \tag{5}$$

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Example: let us consider a parametric u^0 in the form

$$u^0(x) = \mu_1 u_1^0(x) + \mu_2 u_2^0(x) + \mu_3 u_3^0(x); \tag{6}$$

with $\mu = [\mu_1, \mu_2, \mu_3] \in [0,1]^3$ (we denote $\mathscr{P} := [0,1]^3$) and

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$$x = x_0 + u^0(x_0)t. \tag{7}$$

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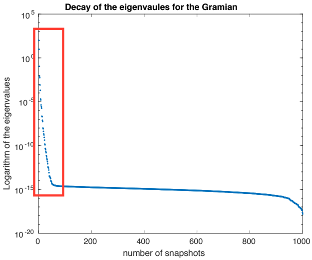
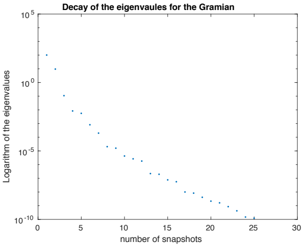
Unknown: $u(x,t) \to x_0 = x_0(x,t;\mu)$.

Numerical test

We take a space $\mathscr{P}_{train} = \mathscr{P}^{1000}$, and a discretization of 10 steps both in time and space.



(a) Global decay of the eigenvalues



(b) Zoom of the red part

The POD provides a space with small dimension X_k and a projector Ψ_k.

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With $k = 20$ POD modes, we proceed the approximation below.

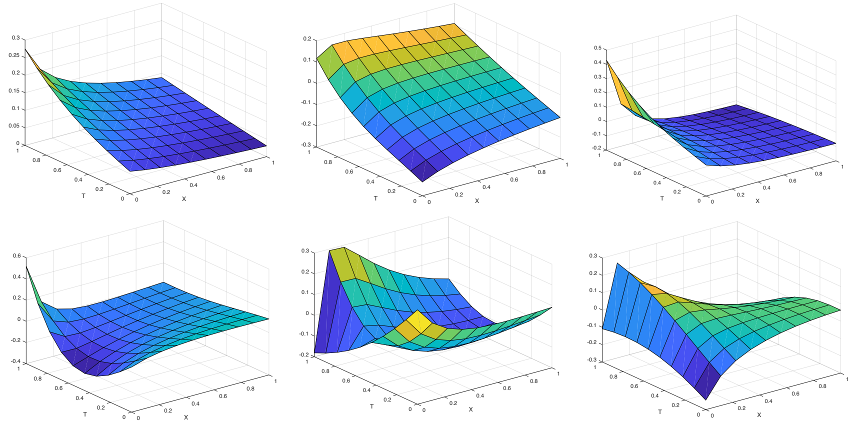

Figure: The first-six POD modes.

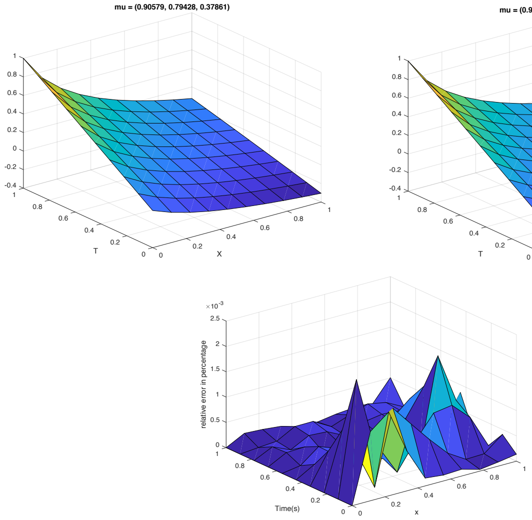

Figure: Compare the exact solution (a) with the approximate solution (b) for x_0 for the same parameters and relative error in percentage (c).

Further test, we now take a space $\mathscr{P}_{train} = \mathscr{P}^{120}$ with a discretization of 50 steps in space and in time.

k	8	15	30	60	120
Accuracy	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
Computation time(s)	5.21	21.81	100.13	334.54	1262.5

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- Mathematical modeling: 2D Saint-Venant model

$$\begin{cases} \partial_t H + div(H\mathbf{u}) = 0, \\ \partial_t H\mathbf{u} + div(H\mathbf{u} \otimes \mathbf{u}) + \frac{1}{2}\nabla g H^2 = -gh\nabla z_b - \mu g H sgn(\mathbf{u}), \end{cases} \tag{8}$$

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- Objectif: Apply the ROM into this model

Real Avalanche

Numerical Avalanche with FreshKiss1D

The 1D Saint-Venant system in the Eulerian form with the coulomb friction:

$$\begin{cases} \partial_t h + \partial_x hu = 0, \\ \partial_t hu + \partial_x (hu^2 + g\frac{h^2}{2}) = -\mu ghsgn(u), \end{cases} \tag{9}$$

We consider the change of variable: $(t, x) \mapsto (\tau, y)$

$$\tau = t \text{ and } y(x,t) = \int_{-\infty}^{x} h(s,t)ds; \tag{10}$$

The 1D Saint-Venant system in the Lagrangian representation:

$$\begin{cases} \partial_\tau \frac{1}{h} - \partial_y u = 0, \\ \partial_\tau u + gh\partial_y h = -\mu gsgn(u), \end{cases} \tag{11}$$

The Gramian matrix with L^2 inner product, $X = (h, hu)$ and $Y = (h, u)$.

$$\|X\|^2 = \int_x (h^2 + (hu)^2)dx \quad \|Y\|^2 = \int_y (h^2 + u^2)dy$$

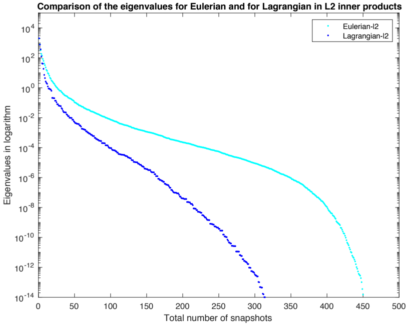

Figure: The eigenvalues in the Eulerian form(cyan) and the Lagrangian form(blue).

The Gramian matrix with energy inner product, $X = (h, \sqrt{h}u)$ and $Y = (\sqrt{h}, u)$.

$$\|X\|^2 = \int_x (h^2 + hu^2)dx \quad \|Y\|^2 = \int_y (h + u^2)dy$$

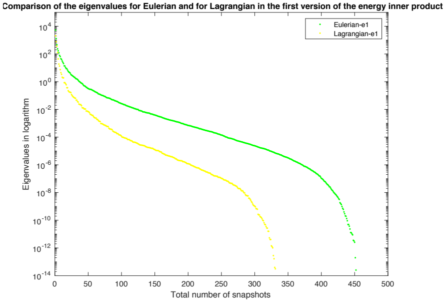

Figure: The eigenvalues in the Eulerian form(green) and the Lagrangian form(yellow).

The Gramian matrix with energy inner product,

$X = \left(\sqrt{\frac{g(h+2z_b)^2 - z_b^2}{2}}, \sqrt{h}u\right)$ and $Y = \left(\sqrt{\frac{g}{2}}(h+2z_b), u\right)$.

$$\int_x hu^2 + \frac{g(h+z_b)^2 - z_b^2}{2}dx \quad \int_y u^2 + \frac{g(h+2z_b)^2}{2}dy$$

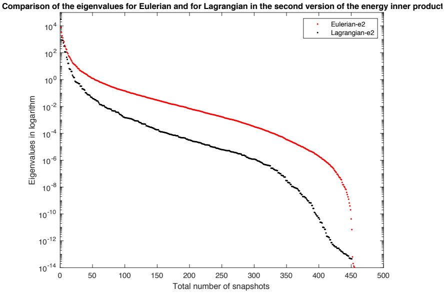

Figure: The eigenvalues in the Eulerian form(red) and the Lagrangian form(black).

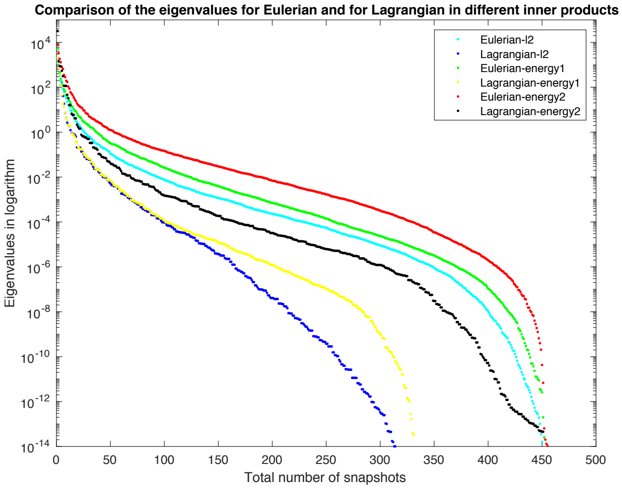

Figure: The eigenvalues in the Eulerian form and the Lagrangian form for different inner products.

Overview

What we have done

1. Studied two ROM methods: POD and DEIM;
2. **Combination of the characteristic method with the ROM method, and apply for Burger's equation**;
3. **Compared the reduction of SV system under two different representations**.

What we may do in the future

1. Complete the RB for the Lagrangian representation with different inner products;
2. Use the ROM method for viscous Burgers' equation.