Domain Decomposition Methods and Applications for Optimal Control Problems

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Overview

History of Domain Decomposition Methods

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★ **Hermann A. Schwarz (1870)**: Über einen Grenzübergang durch alternierendes Verfahren
(en): Over a Boundary transition by alternating method
(fr): Sur un passage de frontière par une procédure alternée

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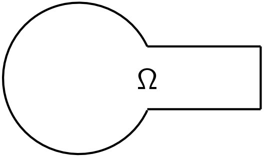

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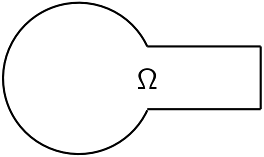

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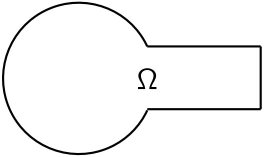
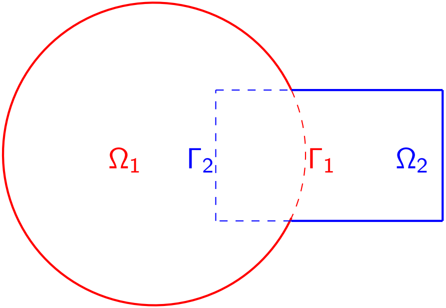


★ **Problem**: existence and uniqueness of (1) in Ω?

★ **Tools**: ~~Sobolev space~~, ~~Lax-Milgram theorem~~, Fourier transform.

Classical Alternating Schwarz Method

Domain: $\Omega := \Omega_1 \cup \Omega_2$



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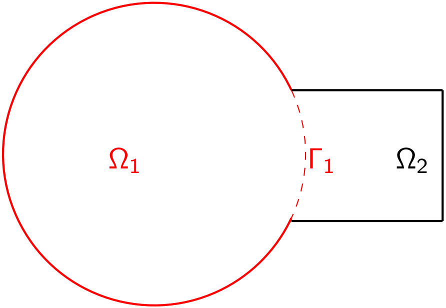

$$\begin{aligned} -\Delta y_1^1 &= f && \text{in } \Omega_1, \\ y_1^1 &= g && \text{on } \partial\Omega \cap \bar{\Omega}_1, \\ y_1^1 &= y_2^0 && \text{on } \Gamma_1 \end{aligned}$$

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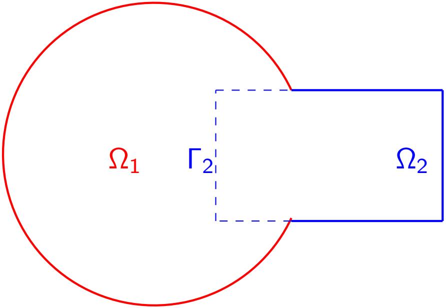

$$
\begin{aligned}
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\end{aligned}
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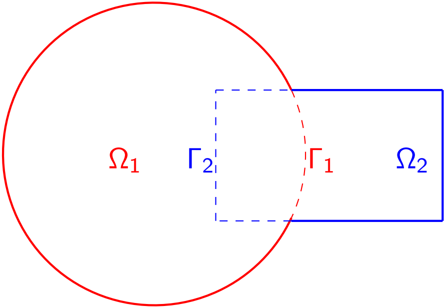

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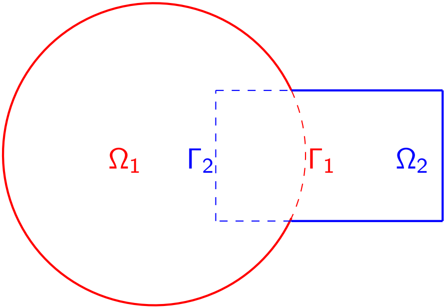

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Convergence: Schwarz proved in 1869 using the maximum principle.

Development of Domain Decomposition Methods

★ High speed computing, parallel computing
  - Overlapping: alternating Schwarz Method, additive Schwarz method, etc.
  - Non-Overlapping: Substructuring Methods (Dirichlet-Neumann, Neumann-Neumann), Balancing Domain Decomposition by Constraint (BDDC), etc.

★ Preconditioners for Krylov, Conjugate Gradient, GMRES, etc

Optimal control

★ **Ingredients**:

- A *system* governed by an ODE/PDE (state y),
- A *control* function u as an input to the system,
- A *target state* $\hat{y}$ as the desired state of the system,
- A *cost functional* J, e.g., cost of u, discrepancy between y and $\hat{y}$, etc.

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★ **Goal**:

- Find the control u^* which minimizes the cost such that the system reaches the desired state.

Overview

Heat equation

★ **Model**:

$$
\begin{aligned}
\partial_t y - \Delta_x y &= u && \text{in } Q, \\
y &= 0 && \text{on } \Sigma, \\
y &= y_0 && \text{on } \Sigma_0,
\end{aligned}
\tag{2}
$$

with the time-space domain $Q := (0, T) \times \Omega$, the lateral boundary $\Sigma := (0, T) \times \partial\Omega$, $\Sigma_0 := \{0\} \times \Omega$ and $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$.

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★ **Problem**:

$$J(y, u) = \frac{1}{2}\|y - \hat{y}\|^2_{L^2(Q)} + \frac{\gamma}{2}\|y(T) - \hat{y}(T)\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{U_{\text{ad}}}, \tag{3}$$

with $\gamma, \nu > 0$.

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★ **Goal**: Find

$$\min J(y, u),$$

subject to the PDE constraint (2).

Optimality system

- Lagrange multiplier approach:

$$L(y, \lambda, u) = J(y, u) + \langle \lambda, \partial_t y - \Delta_x y - u \rangle,$$

λ is the Lagrange multiplier or adjoint state.

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- Integration by parts

$$\begin{aligned}\langle \lambda, \partial_t y - \Delta_x y - u \rangle = & - \langle \partial_t \lambda, y \rangle + (\lambda(T), y(T)) - (\lambda(0), y(0)) \\ & - \langle \Delta_x \lambda, y \rangle - \int_\Sigma \partial_n y \lambda + \int_\Sigma y \partial_n \lambda \\ & - \langle \lambda, u \rangle.\end{aligned}$$

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- Optimality condition:

$$\partial_u L(y,\lambda,u) = 0 \quad \Rightarrow \quad -\lambda + \nu u = 0.$$

with $U_{\text{ad}} := L^2(Q)$.

Optimality system

- First-order optimality system (forward-backward):

$$\begin{aligned} \partial_t y - \Delta_x y &= u, & \partial_t \lambda + \Delta_x \lambda &= y - \hat{y}, \\ y(\cdot, x) &= 0, & \lambda(\cdot, x) &= 0, \\ y(0, \cdot) &= y_0, & \lambda(T, \cdot) &= \gamma(y(T, \cdot) - \hat{y}(T, \cdot)), \end{aligned}$$

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- Semi-discretization version:

$$\begin{aligned} \dot{y} + Ay &= \nu^{-1}\lambda, & \dot{\lambda} - A^T\lambda &= y - \hat{y}, \\ y(0) &= 0, & \lambda(T) &= \gamma(y(T) - \hat{y}(T)), \end{aligned}$$

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- $A = A^T \Rightarrow A = QDQ^T$ with $Q^TQ = I$ and $D = \mathrm{diag}(d_1, \ldots, d_m)$.

$$\begin{aligned} \dot{\tilde{y}} + D\tilde{y} &= \nu^{-1}\tilde{\lambda}, & \dot{\tilde{\lambda}} - D\tilde{\lambda} &= \tilde{y} - \tilde{\hat{y}}, \\ \tilde{y}(0) &= 0, & \tilde{\lambda}(T) &= \gamma(\tilde{y}(T) - \tilde{\hat{y}}(T)), \end{aligned}$$

  with $\tilde{y} = Q^Ty$, $\tilde{\hat{y}} = Q^T\hat{y}$ and $\tilde{\lambda} = Q^T\lambda$.

Optimality system

- m independent 2×2 systems:

$$\begin{cases} \begin{pmatrix} \dot{y} \\ \dot{\lambda} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{y} \end{pmatrix}, \\ y(0) = y_0, \\ \lambda(T) = \gamma(y(T) - \hat{y}(T)), \end{cases}$$

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- Second-order ODE

$$\begin{cases} \nu\ddot{y} - (\nu d_i^2 + 1)y = -\hat{y}, \\ y(0) = y_0, \\ \nu\dot{y}(T) + (\nu d_i + \gamma)y(T) = \gamma\hat{y}(T), \end{cases}$$

Dirichlet-Neumann (Bjørstad, Widlund 1986)

★ **Domain**: $\Omega_1 := (0, \Gamma), \Omega_2 := (\Gamma, T)$ where Γ is the interface

$$\begin{aligned} \nu\ddot{y}_1^k - (\nu d_i^2 + 1)y_1^k + \hat{y} &= 0, & \nu\ddot{y}_2^k - (\nu d_i^2 + 1)y_2^k + \hat{y} &= 0, \\ y_1^k(0) &= y_0, & \nu\dot{y}_2^k(T) + (\nu d_i + \gamma)y_2^k(T) &= \gamma\hat{y}(T), \\ y_1^k(\Gamma) &= y_2^{k-1}(\Gamma), & \dot{y}_2^k(\Gamma) &= \dot{y}_1^k(\Gamma). \end{aligned}$$

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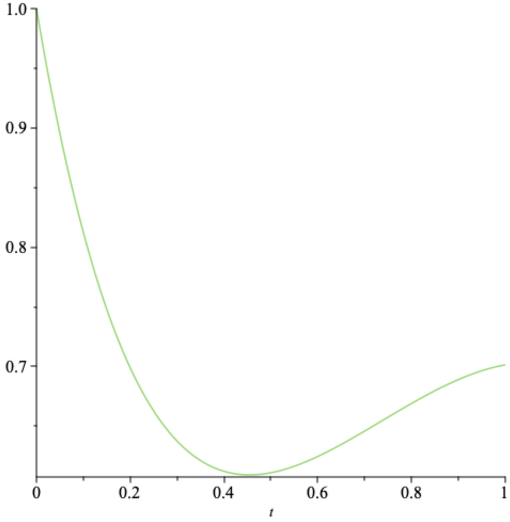
★ **Test**: $d_i = 0.5$, $\nu = 0.1$, $\gamma = 0.3$, $T = 1$, $\Gamma = \frac{2}{3}$, $y_0 = 1$,
$\hat{y}(t) = \sin(t)$, $y^0(\Gamma) = 0$.

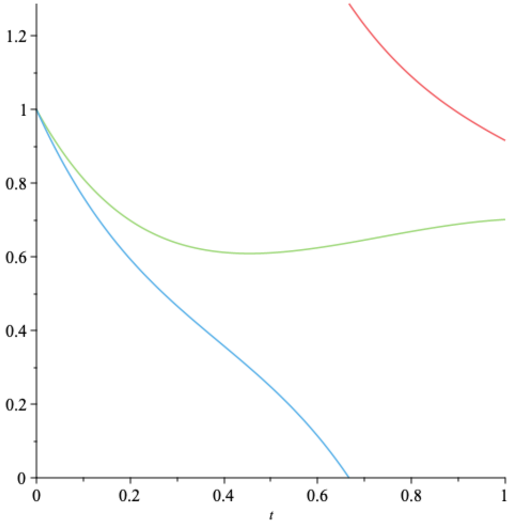
Dirichlet-Neumann

Exact solution:



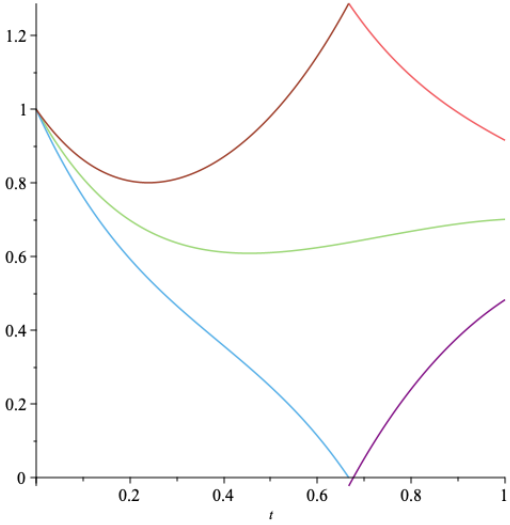
Dirichlet-Neumann

First iteration:



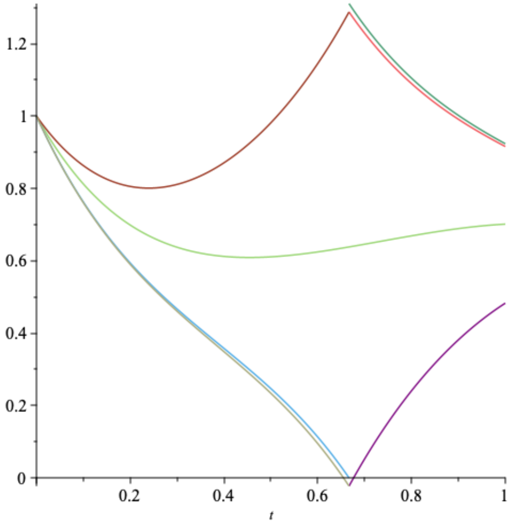
Dirichlet-Neumann

Second iteration:

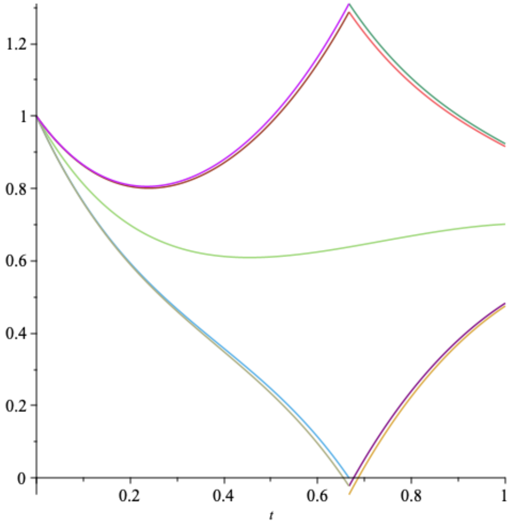


Dirichlet-Neumann

Third iteration:



Fourth iteration:



Error Analysis

- Error equation for $e_j^k := y - y_j^k$

$$
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\nu \ddot{e}_1^k - (\nu d_i^2 + 1) e_1^k &= 0, & \nu \ddot{e}_2^k - (\nu d_i^2 + 1) e_2^k &= 0,\\
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- Solution:

$$e_1^k(t) = A^k \sinh(\alpha t), \quad e_2^k(t) = B^k \left(\cosh\left(\alpha(T-t)\right) + \beta \sinh\left(\alpha(T-t)\right)\right),$$

with $\alpha := \sqrt{\frac{\nu d_i^2+1}{\nu}}$ and $\beta := \frac{\nu d_i + \gamma}{\sqrt{\nu^2\, d_i^2 + \nu}}$.

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- Coefficients:

$$A^k = \frac{e_2^{k-1}(\Gamma)}{\sinh(\alpha\Gamma)}, \quad B^k = -\frac{e_2^{k-1}(\Gamma)\coth(\alpha\Gamma)}{\sinh(\alpha(T-\Gamma))+\beta\cosh(\alpha(T-\Gamma))}.$$

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- Convergence factor:

$$e_2^k(\Gamma) = -e_2^{k-1}(\Gamma)\frac{\cosh(\alpha(T-\Gamma)) + \beta\sinh(\alpha(T-\Gamma))}{\sinh(\alpha(T-\Gamma)) + \beta\cosh(\alpha(T-\Gamma))}\coth(\alpha\Gamma).$$

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$$e_1^k(t) = A^k \sinh(\alpha t), \quad e_2^k(t) = B^k \left(\cosh\left(\alpha(T-t)\right) + \beta \sinh\left(\alpha(T-t)\right)\right),$$

with $\alpha := \sqrt{\frac{\nu d_i^2 + 1}{\nu}}$ and $\beta := \frac{\nu d_i + \gamma}{\sqrt{\nu^2 d_i^2 + \nu}}$.

- Coefficients:

$$A^k = \frac{e_2^{k-1}(\Gamma)}{\sinh(\alpha\Gamma)}, \quad B^k = -\frac{e_2^{k-1}(\Gamma)\coth(\alpha\Gamma)}{\sinh(\alpha(T-\Gamma)) + \beta\cosh(\alpha(T-\Gamma))}.$$

- Convergence factor:

$$\rho_{\text{DN}} := \frac{\cosh\left(\alpha(T-\Gamma)\right) + \beta \sinh\left(\alpha(T-\Gamma)\right)}{\sinh\left(\alpha(T-\Gamma)\right) + \beta \cosh\left(\alpha(T-\Gamma)\right)} \coth(\alpha\Gamma).$$

Dirichlet-Neumann with relaxation

★ **Domain**: $\Omega_1 := (0, \Gamma), \Omega_2 := (\Gamma, T)$ where Γ is the interface

$$\nu \ddot{y}_1^k - (\nu d_i^2 + 1) y_1^k + \hat{y} = 0, \qquad \nu \ddot{y}_2^k - (\nu d_i^2 + 1) y_2^k + \hat{y} = 0,$$
$$y_1^k(0) = y_0, \quad \nu \dot{y}_2^k(T) + (\nu d_i + \gamma) y_2^k(T) = \gamma \hat{y}(T),$$
$$y_1^k(\Gamma) = y_\Gamma^{k-1}, \qquad \dot{y}_2^k(\Gamma) = \dot{y}_1^k(\Gamma).$$

with $y_\Gamma^k := (1 - \theta) y_\Gamma^{k-1} + \theta y_2^k(\Gamma)$.

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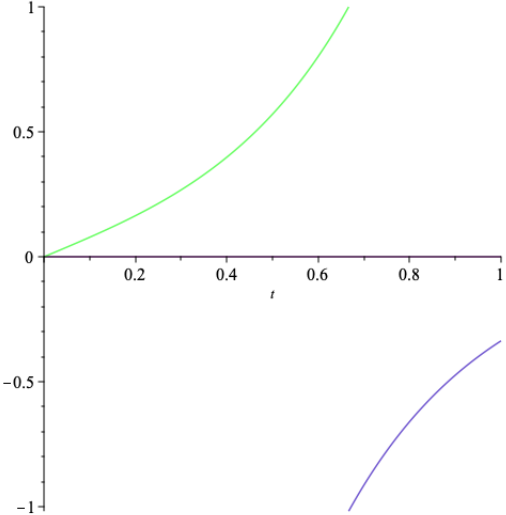
★ **Test**: $d_i = 0.5$, $\nu = 0.1$, $\gamma = 0.3$, $T = 1$, $\Gamma = \frac{2}{3}$, $y_0 = 0$, $\hat{y} = 0$, $y_\Gamma^0 = 1$, $\theta = \frac{1}{2}$.

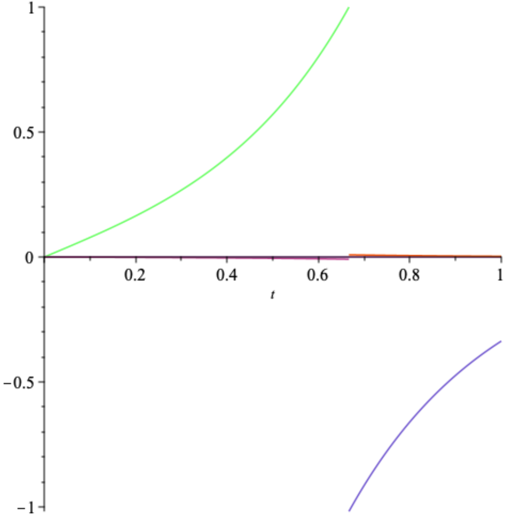
Dirichlet-Neumann with relaxation

First iteration:



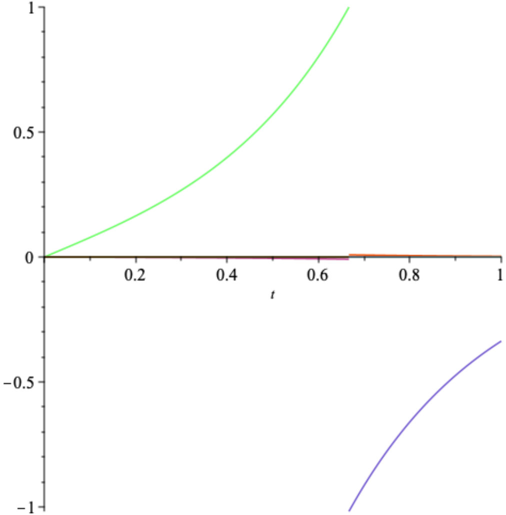
Dirichlet-Neumann with relaxation

Second iteration:



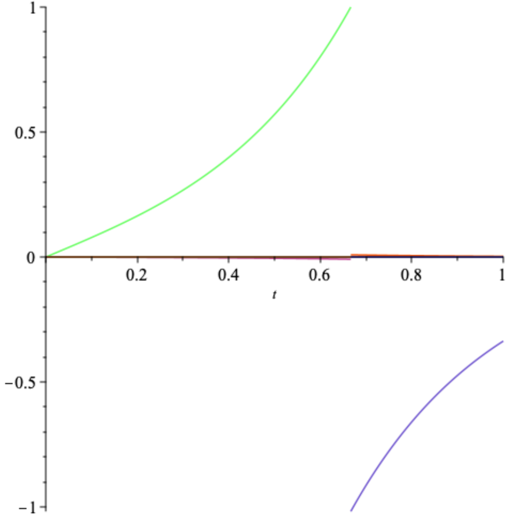
Dirichlet-Neumann with relaxation

Third iteration:



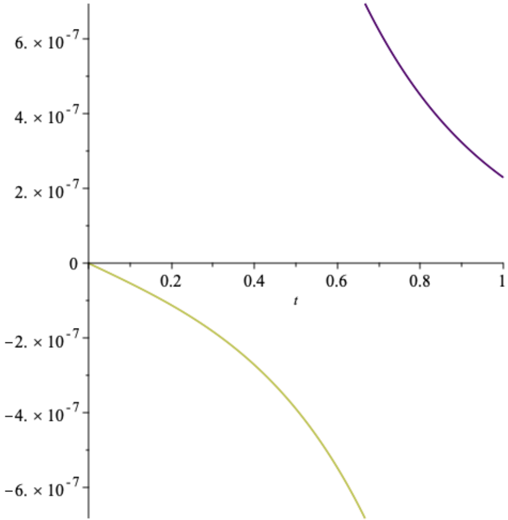
Dirichlet-Neumann with relaxation

Fourth iteration:



Dirichlet-Neumann with relaxation

Fourth iteration:



- Error equation for $e_j^k := y - y_j^k$

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- Convergence factor:

$$e_\Gamma^k = (1-\theta)e_\Gamma^{k-1} + \theta e_\Gamma^{k-1}\frac{\cosh(\alpha(T-\Gamma)) + \beta\sinh(\alpha(T-\Gamma))}{\sinh(\alpha(T-\Gamma)) + \beta\cosh(\alpha(T-\Gamma))}\coth(\alpha\Gamma).$$

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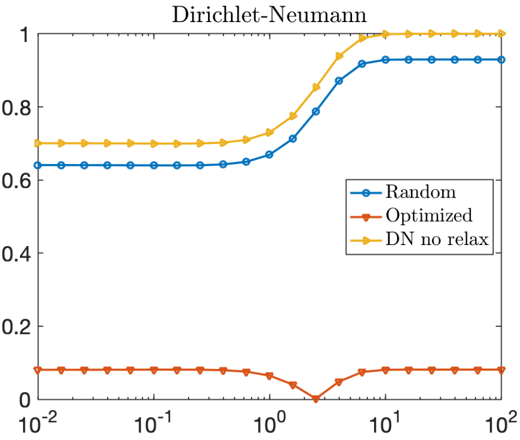
- Coefficients:

$$A^k = \frac{e_\Gamma^{k-1}}{\sinh(\alpha\Gamma)}, \quad B^k = -\frac{e_\Gamma^{k-1} \coth(\alpha\Gamma)}{\sinh\left(\alpha(T-\Gamma)\right) + \beta \cosh\left(\alpha(T-\Gamma)\right)}$$

- Convergence factor:

$$\rho_{\text{DNR}} := 1 - \theta \frac{\cosh\left(\alpha T\right) + \beta \sinh\left(\alpha T\right)}{\sinh(\alpha\Gamma)\left(\sinh\left(\alpha(T-\Gamma)\right) + \beta \cosh\left(\alpha(T-\Gamma)\right)\right)}.$$

Convergence factor



Neumann-Neumann (Bourgat, Glowinski, Tallec, Vidrascu 1989)

★ **Domain**: $\Omega_1 := (0, \Gamma), \Omega_2 := (\Gamma, T)$ where Γ is the interface, for $j = 1, 2$

$$\begin{aligned} \nu \ddot{y}_j^k - (\nu d_i^2 + 1) y_j^k + \hat{y} &= 0, & \nu \ddot{\psi}_j^k - (\nu d_i^2 + 1) \psi_j^k &= 0, \\ y_1^k(0) &= y_0, & \psi_1^k(0) &= 0, \\ \nu \dot{y}_2^k(T) + (\nu d_i + \gamma) y_2^k(T) &= \gamma \hat{y}(T), & \psi_2^k(T) &= 0, \\ y_j^k(\Gamma) &= y_\Gamma^{k-1}, & \partial_{n_j} \psi_j^k &= \partial_{n_1} y_1^k + \partial_{n_2} y_2^k. \end{aligned}$$

with $y_\Gamma^k := y_\Gamma^{k-1} - \theta \left(\psi_1^k(\Gamma) + \psi_2^k(\Gamma) \right)$.

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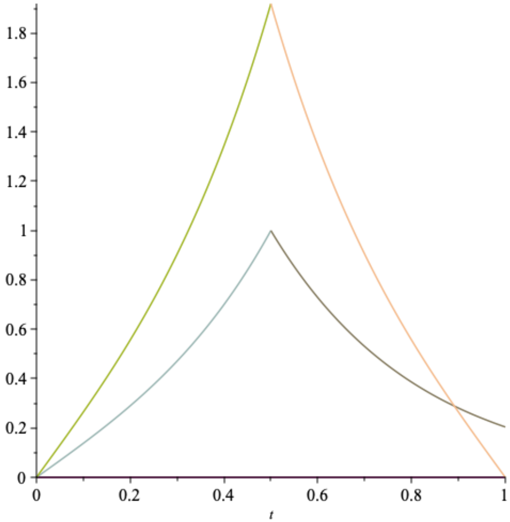
★ **Test**: $d_i = 0.5$, $\nu = 0.1$, $\gamma = 0.3$, $T = 1$, $\Gamma = \frac{1}{2}$, $y_0 = 0$, $\hat{y} = 0$, $y^0(\Gamma) = 1$, $\theta = \frac{1}{4}$.

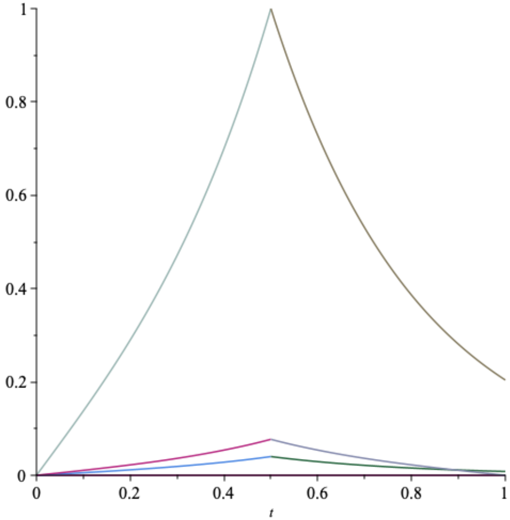
Neumann-Neumann

First iteration:



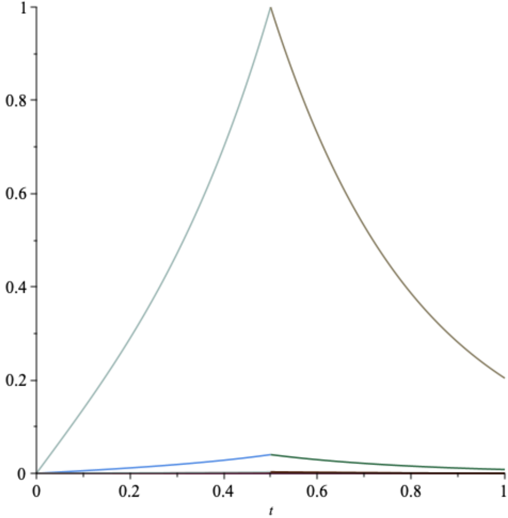
Neumann-Neumann

Second iteration:



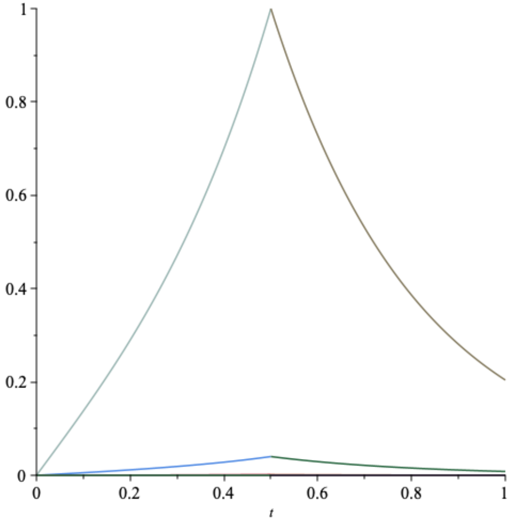
Neumann-Neumann

Third iteration:



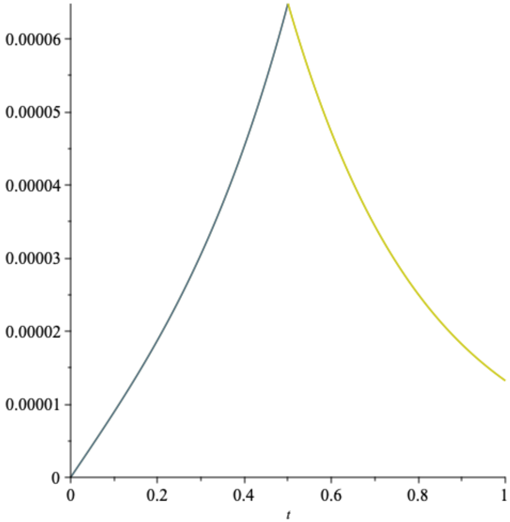
Neumann-Neumann

Fourth iteration:



Neumann-Neumann

Fourth iteration:



- Error equation for $e_j^k := y - y_j^k$

$$
\begin{aligned}
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$$
\psi_1^k(t) = C^k\sinh(\alpha t), \quad \psi_2^k(t) = D^k\sinh\left(\alpha(T-t)\right)e^{-\alpha T}.
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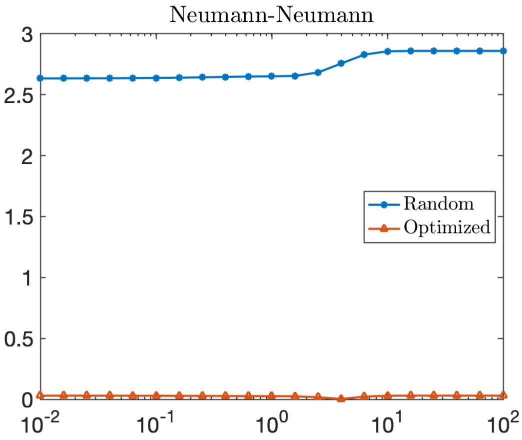
- Convergence factor:

$$
\rho_{\mathrm{NN}} := 1 - \theta\tfrac{\sinh(\alpha T)}{\cosh(\alpha\Gamma)\cosh(\alpha(T-\Gamma))}\tfrac{\cosh(\alpha T)+\beta\sinh(\alpha T)}{\sinh(\alpha\Gamma)(\cosh(\alpha(T-\Gamma))+\beta\sinh(\alpha(T-\Gamma)))}.
$$

Convergence factor



Overview

Poisson's equation

★ **Model**:

$$\begin{aligned} -\Delta y &= u \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4}$$

with $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$.

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★ **Problem**:

$$J(y, u) = \frac{1}{2} \int_\Omega |y(x) - \hat{y}(x)|^2 \, \mathrm{d}x + \frac{\nu}{2} \|u\|^2_{U_{\text{ad}}}, \tag{5}$$

with $\nu > 0$.

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with $\nu > 0$.

★ **Goal**: Find

$$\min J(y, u),$$

subject to the PDE constraint (2).

L^2 Regularization

- Lagrange multiplier approach:

$$L(y, \lambda, u) = J(y, u) + \langle \lambda, -\Delta y - u \rangle,$$

λ is the Lagrange multiplier or adjoint state.

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$$\partial_\lambda L(y, \lambda, u) = 0 \quad \Rightarrow \quad (4).$$

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- Integration by parts

$$\langle \lambda, -\Delta y - u \rangle = -\langle \Delta \lambda, y \rangle - \int_{\partial\Omega} \partial_n y \lambda + \int_{\partial\Omega} y \partial_n \lambda - \langle \lambda, u \rangle.$$

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- Optimality condition:

$$\partial_u L(y, \lambda, u) = 0 \quad \Rightarrow \quad -\lambda + \nu u = 0.$$

  with $U_{\text{ad}} := L^2(\Omega)$.

Optimality system

★ First-order optimality system (forward-backward):

$$
\begin{aligned}
-\Delta y &= \nu^{-1}\lambda \text{ in } \Omega & \qquad -\Delta\lambda &= y - \hat{y} & \quad &\text{in } \Omega,\\
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★ Bi-Laplacian:

$$\begin{aligned} \nu\Delta^2 y &= y - \hat{y} \text{ in } \Omega \\ y &= 0 \text{ on } \partial\Omega \\ \Delta y &= 0, \text{ on } \partial\Omega \end{aligned}$$

H^{-1} Regularization

- Lagrange multiplier approach:

$$L(y, \lambda, u) = J(y, u) - \langle \lambda, -\Delta y - u \rangle,$$

  λ is the Lagrange multiplier or adjoint state.
- Derive first-order optimality system formally.
- Primal problem:

$$\partial_\lambda L(y, \lambda, u) = 0 \quad \Rightarrow \quad (4).$$

- Adjoint problem:

$$\partial_y L(y, \lambda, u) = 0 \quad \Rightarrow \quad \begin{array}{cl} \Delta \lambda = y - \hat{y} & \text{in } \Omega, \\ \lambda = 0 & \text{on } \partial\Omega, \end{array}$$

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$$\partial_u L(y, \lambda, u) = 0,$$

  definition of $\|u\|^2_{H^{-1}(\Omega)}$.

Energy norm

- A linear operator $\mathcal{H} : H^{-1}(\Omega) \to H_0^1(\Omega) \subset L^2(\Omega)$ such that $\mathcal{H}u$ is the unique solution of the variational problem related to (4)

$$\int_\Omega \nabla \mathcal{H}u(x) \cdot \nabla v(x) \, \mathrm{d}x = \langle u, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega), \tag{6}$$

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- The norm in $H^{-1}(\Omega)$ which is equivalent to the energy norm

$$\|u\|^2_{H^{-1}(\Omega)} := \langle u, \mathcal{H}u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \|\nabla y\|^2_{L^2(Q)}. \tag{7}$$

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- The norm in $H^{-1}(\Omega)$ which is equivalent to the energy norm

$$\|u\|_{H^{-1}(\Omega)}^2 := \langle u, \mathcal{H}u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \|\nabla y\|_{L^2(Q)}^2. \tag{7}$$

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- Identity: $y = \mathcal{H}u$.
- Reduced cost functional:

$$\tilde{J}(u) = \frac{1}{2}\langle \mathcal{H}^*\mathcal{H}u, u \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - \langle \mathcal{H}^* \hat{y}, u \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} + \frac{1}{2}\|\hat{y}\|^2_{L^2(\Omega)} + \frac{\nu}{2}\langle \mathcal{H}u, u \rangle_{H_0^1(\Omega), H^{-1}(\Omega)}$$

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- Lagrange multiplier approach:

$$L(y, \lambda, u) = J(y, u) - \langle \lambda, \Delta y + u \rangle,$$

λ is the Lagrange multiplier or adjoint state.

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using the definition of $\|u\|^2_{H^{-1}(\Omega)}$.

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coupled with $\lambda + \nu \mathcal{H} u = 0$.

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★ Reduction:

Optimality system

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★ Reduction:

- $y = \mathcal{H}u$,
- $\lambda + \nu\mathcal{H}u = 0$,
- $-\nu\Delta y + y = \hat{y}$ in Ω, $\quad y = 0$ on $\partial\Omega$.

Error Analysis (DN)

- Error equation for $e_j^k := y - y_j^k$

$$\nu \ddot{e}_1^k - e_1^k = 0, \quad e_1^k(0) = 0, \quad e_1^k(\Gamma) = e_2^{k-1}(\Gamma),$$
$$\nu \ddot{e}_2^k - e_2^k = 0, \quad e_2^k(1) = 0, \quad \dot{e}_2^k(\Gamma) = \dot{e}_1^k(\Gamma).$$

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- Solution:

$$e_1^k(x) = A^k \sinh(\sqrt{\nu^{-1}} x), \quad e_2^k(x) = B^k \sinh\left(\sqrt{\nu^{-1}}(1-x)\right) e^{-\sqrt{\nu^{-1}}}$$

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$$A^k = \frac{e_2^{k-1}(\Gamma)}{\sinh(\sqrt{\nu^{-1}}\Gamma)}, \quad B^k = -\frac{e_2^{k-1}(\Gamma)\coth(\sqrt{\nu^{-1}}\Gamma)}{\cosh\left(\sqrt{\nu^{-1}}(1-\Gamma)\right)} e^{\sqrt{\nu^{-1}}}$$

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- Convergence factor:

$$e_2^k(\Gamma) = -e_2^{k-1}(\Gamma) \tanh\left(\sqrt{\nu^{-1}}(1-\Gamma)\right) \coth\left(\sqrt{\nu^{-1}}\Gamma\right).$$

Error Analysis (DNR)

- Error equation for $e_j^k := y - y_j^k$

$$\nu \ddot{e}_1^k - e_1^k = 0, \quad e_1^k(0) = 0, \quad e_1^k(\Gamma) = e_\Gamma^{k-1},$$
$$\nu \ddot{e}_2^k - e_2^k = 0, \quad e_2^k(1) = 0, \quad \dot{e}_2^k(\Gamma) = \dot{e}_1^k(\Gamma).$$

with $e_\Gamma^k := (1-\theta)e_\Gamma^{k-1} + \theta e_2^k(\Gamma)$, $\theta \in (0,1)$.

- Solution:

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- Convergence factor:

$$\rho_{\text{DNR}} := 1 - \theta\left[1 + \tanh\left(\sqrt{\nu^{-1}}(1-\Gamma)\right)\coth\left(\sqrt{\nu^{-1}}\Gamma\right)\right].$$

Error Analysis (NN)

- Error equation for $e_j^k := y - y_j^k$:

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$$\nu \ddot{\psi}_j^k - \psi_j^k = 0, \quad \psi_1^k(0) = 0, \quad \psi_2^k(1) = 0, \quad \partial_{n_j} \psi_j^k = \partial_{n_1} e_1^k + \partial_{n_2} e_2^k.$$

with $e_\Gamma^k := e_\Gamma^{k-1} - \theta \left(\psi_1^k(\Gamma) + \psi_2^k(\Gamma) \right)$, $\theta \in (0,1)$.

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- Solution:

$$e_1^k(x) = e_\Gamma^{k-1} \frac{\sinh(\sqrt{\nu^{-1}} x)}{\sinh(\sqrt{\nu^{-1}} \Gamma)}, \quad e_2^k(x) = e_\Gamma^{k-1} \frac{\sinh\left(\sqrt{\nu^{-1}}(1-x)\right)}{\sinh\left(\sqrt{\nu^{-1}}(1-\Gamma)\right)}.$$

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- Correction:

$$\psi_1^k(x) = e_\Gamma^{k-1} \frac{\sinh(\sqrt{\nu^{-1}} x)}{\cosh(\sqrt{\nu^{-1}} \Gamma)} \left(\coth(\sqrt{\nu^{-1}} \Gamma) + \coth(\sqrt{\nu^{-1}}(1-\Gamma)) \right),$$
$$\psi_2^k(x) = e_\Gamma^{k-1} \frac{\sinh\left(\sqrt{\nu^{-1}}(1-x)\right)}{\cosh(\sqrt{\nu^{-1}}(1-\Gamma))} \left(\coth(\sqrt{\nu^{-1}} \Gamma) + \coth(\sqrt{\nu^{-1}}(1-\Gamma)) \right).$$

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- Convergence factor:

$$\rho_{\text{NN}} := 1 - \theta \Big(\tanh(\sqrt{\nu^{-1}} \Gamma) + \tanh\left(\sqrt{\nu^{-1}}(1-\Gamma) \right) \Big) \left(\coth(\sqrt{\nu^{-1}} \Gamma) + \coth(\sqrt{\nu^{-1}}(1-\Gamma)) \right).$$

Overview

Conclusion

★ Parabolic optimal control under L^2 regularization

- Gander and Kwok, *Schwarz Methods for the Time-Parallel Solution of Parabolic Control Problems*, 2016.

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★ Dirichlet-Neumann method and Neumann-Neumann method

★ Error analysis and convergence factor