Non-overlapping domain decomposition methods for parabolic control problems

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Optimal control

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★ **Ingredients**:

- A *system* governed by an ODE/PDE (state y),
- A *control* function u as an input to the system,
- A *target state* $\hat{y}$ as the desired state of the system,
- A *cost functional* J, e.g., cost of u, discrepancy between y and $\hat{y}$, etc.

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★ **Goal**:

- Find the control $u^\star$ which minimizes the cost such that the system reaches the desired state.

Example 1

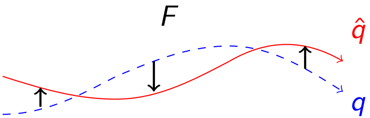
Problem: Compute the force of thrust F

$$\min_{F \in U_{\text{ad}}} \frac{1}{2}\|F\|^2_{U_{\text{ad}}} + \frac{1}{2}\int_0^T |q(t) - \hat{q}(t)|^2 \mathrm{d}t,$$

subject to

$$\ddot{q} = -\frac{q}{|q|^3} + \frac{F}{m}, \qquad \text{in } (0, T),$$

with m the mass of the satellite.



Example 2

Problem: Compute the bottom topography z_b

$$\max_{z_b \in U_{\mathrm{ad}}} \mathcal{P}(z_b, X, I),$$

subject to

$$\dot{X} = f(X, I),$$

with I the light perceived.

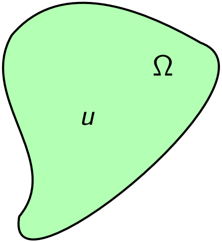
Example 3

Problem: Compute the heat source u

$$J(y,u) = \frac{1}{2}\|y-\hat{y}\|^2_{L^2(Q)} + \frac{\nu}{2}\|u\|^2_{U_{\mathrm{ad}}},$$

subject to

$$\partial_t y - \Delta_x y = u, \quad \text{in } (0,T)\times\Omega.$$



Heat equation

★ **Model**:

$$\begin{aligned} \partial_t y - \Delta_x y &= u && \text{in } Q, \\ y &= 0 && \text{on } \Sigma, \\ y &= y_0 && \text{on } \Sigma_0, \end{aligned} \tag{1}$$

$Q := (0, T) \times \Omega$, $\Sigma := (0, T) \times \partial\Omega$, $\Sigma_0 := \{0\} \times \Omega$ and $\Omega \subset \mathbb{R}^n$.

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with $\nu > 0$.

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subject to the PDE constraint (1).

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★ **Approach**: Lagrange multiplier λ

$$L(y, \lambda, u) = J(y, u) + \langle \lambda, \partial_t y - \Delta_x y - u \rangle.$$

Optimality system

$$L(y, \lambda, u) = \frac{1}{2}\|y - \hat{y}\|^2_{L^2(Q)} + \frac{\nu}{2}\|u\|^2_{U_{\text{ad}}} + \langle \lambda, \partial_t y - \Delta_x y - u \rangle,$$

- Primal problem:

$$\partial_\lambda L(y, \lambda, u) = 0 \quad \Rightarrow \quad \partial_t y - \Delta_x y = u.$$

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- Integration by parts

$$\begin{aligned}\langle \lambda, \partial_t y - \Delta_x y\rangle = &- \langle \partial_t \lambda, y\rangle + (\lambda(T), y(T)) - (\lambda(0), y(0)) \\ &- \langle \Delta_x \lambda, y\rangle - \int_\Sigma \partial_n y \lambda + \int_\Sigma y \partial_n \lambda.\end{aligned}$$

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- Optimality condition:

$$\partial_u L(y, \lambda, u) = 0 \quad \Rightarrow \quad -\lambda + \nu u = 0.$$

with $U_{\text{ad}} := L^2(Q)$.

Optimality system

- First-order optimality system (forward-backward):

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\partial_t y - \Delta_x y &= u \quad \text{in } Q, & -\partial_t \lambda - \Delta_x \lambda &= y - \hat{y} \quad \text{in } Q,\\
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- Semi-discretization version:

$$
\begin{aligned}
\dot{y} + Ay &= \nu^{-1}\lambda, & \dot{\lambda} - A^T\lambda &= y - \hat{y},\\
y(0) &= y_0, & \lambda(T) &= 0,
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- $A = A^T \Rightarrow A = QDQ^T$ with $Q^T Q = I$ and $D = \mathrm{diag}(d_1, \dots, d_m)$.

$$
\begin{aligned}
\dot{\tilde{y}} + D\tilde{y} &= \nu^{-1}\tilde{\lambda}, & \dot{\tilde{\lambda}} - D\tilde{\lambda} &= \tilde{y} - \tilde{\hat{y}},\\
\tilde{y}(0) &= 0, & \tilde{\lambda}(T) &= 0,
\end{aligned}
$$

with $\tilde{y} = Q^T y$, $\tilde{\hat{y}} = Q^T\hat{y}$ and $\tilde{\lambda} = Q^T\lambda$.

Optimality system

- m independent 2×2 systems:

$$
\begin{cases}
\begin{pmatrix} \dot{y} \\ \dot{\lambda} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{y} \end{pmatrix}, \\
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- Second-order ODE

$$\begin{cases} \nu \ddot{y} - (\nu d_i^2 + 1) y = -\hat{y}, \\ \hfill y(0) = y_0, \\ \hfill \dot{y}(T) + d_i y(T) = 0, \end{cases}$$

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- Previous work

Gander and Kwok, *Schwarz Methods for the Time-Parallel Solution of Parabolic Control Problems*, 2016.

Example: $\Omega_1 := (0, \Gamma), \Omega_2 := (\Gamma, 1)$ with the interface $\Gamma = 1/2$.

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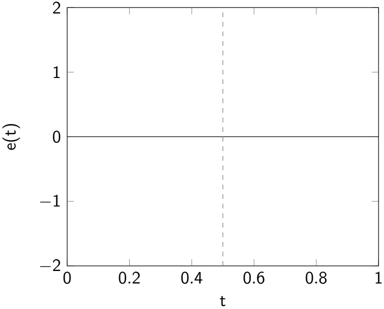

Dirichlet-Neumann (Bjørstad, Widlund 1986)

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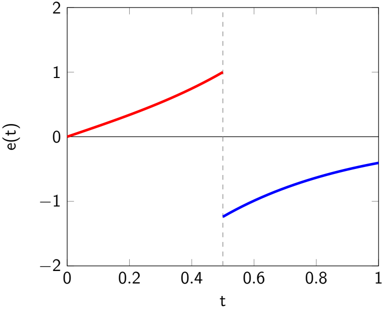

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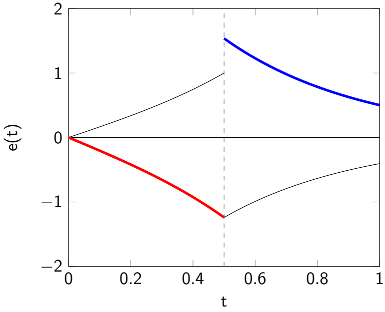

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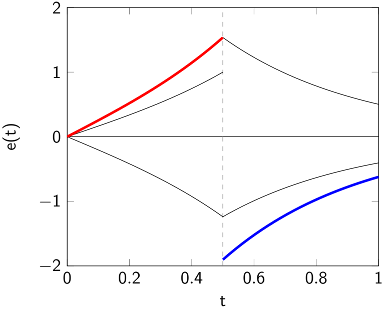

Convergence Analysis

- Error equation for $e_j^k := y - y_j^k$

$$\begin{aligned} \nu \ddot{e}_1^k - (\nu d_i^2 + 1) e_1^k &= 0, & \nu \ddot{e}_2^k - (\nu d_i^2 + 1) e_2^k &= 0, \\ e_1^k(0) &= 0, & \dot{e}_2^k(T) + d_i e_2^k(T) &= 0, \\ e_1^k(\Gamma) &= {\color{red} e_2^{k-1}(\Gamma)}, & \dot{e}_2^k(\Gamma) &= {\color{blue} \dot{e}_1^k(\Gamma)}. \end{aligned}$$

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\end{aligned}
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- Solution:

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\begin{aligned}
e_1^k(t) &= A^k \sinh(\alpha t),\\
e_2^k(t) &= B^k \big[\cosh\left(\alpha(T-t)\right) + \beta \sinh\left(\alpha(T-t)\right)\big],
\end{aligned}
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with $\alpha := \sqrt{\frac{\nu d_i^2 + 1}{\nu}}$ and $\beta := \frac{\nu d_i}{\sqrt{\nu^2 d_i^2 + \nu}}$.

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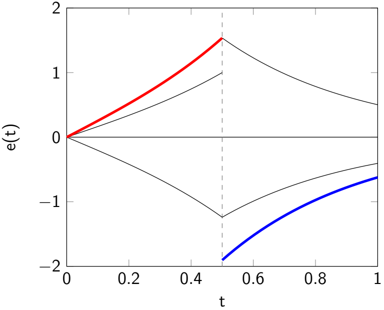
- Convergence factor:

$$e_2^k(\Gamma) = -e_2^{k-1}(\Gamma) \underbrace{\frac{\cosh\left(\alpha(T-\Gamma)\right) + \beta \sinh\left(\alpha(T-\Gamma)\right)}{\sinh\left(\alpha(T-\Gamma)\right) + \beta \cosh\left(\alpha(T-\Gamma)\right)} \coth(\alpha\Gamma)}_{\rho_{DN}}.$$

- Previous example:



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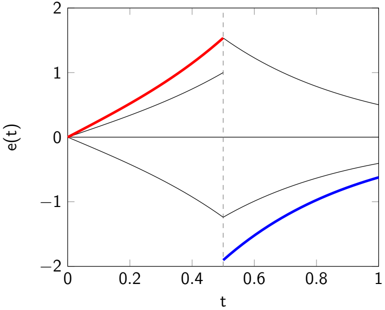

- For $\Gamma = \frac{T}{2}$

$$\rho_{DN}\big|_{\Gamma=\frac{T}{2}} = \frac{\cosh\left(\alpha\frac{T}{2}\right) + \beta\sinh\left(\alpha\frac{T}{2}\right)}{\sinh\left(\alpha\frac{T}{2}\right) + \beta\cosh\left(\alpha\frac{T}{2}\right)} \cdot \frac{\cosh(\alpha\frac{T}{2})}{\sinh(\alpha\frac{T}{2})}$$

$$= 1 + \frac{1}{\sinh^2(\alpha\frac{T}{2}) + \beta\cosh(\alpha\frac{T}{2})\sinh(\alpha\frac{T}{2})}.$$

Dirichlet-Neumann with relaxation

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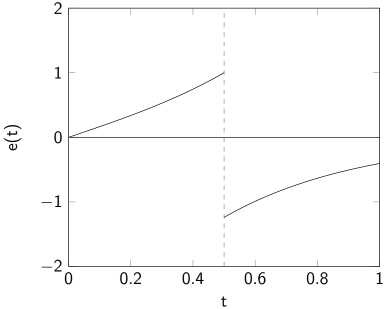
$$\begin{aligned} \nu\ddot{e}_1^k - (\nu d_i^2 + 1)e_1^k &= 0, & \nu\ddot{e}_2^k - (\nu d_i^2 + 1)y_2^k &= 0,\\ e_1^k(0) &= 0, & \dot{e}_2^k(T) + d_i e_2^k(T) &= 0,\\ e_1^k(\Gamma) &= {\color{red}e_\Gamma^{k-1}}, & \dot{e}_2^k(\Gamma) &= \dot{e}_1^k(\Gamma). \end{aligned}$$

with $y_\Gamma^k := (1-{\color{red}\theta})y_\Gamma^{k-1} + {\color{red}\theta} y_2^k(\Gamma)$.



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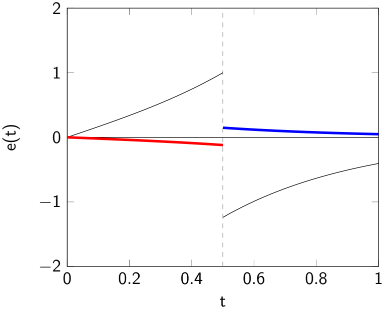

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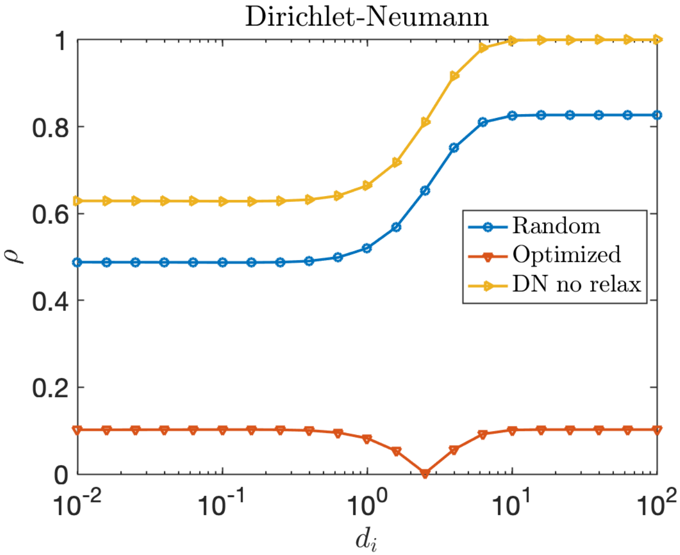
- Convergence factor:

$$
\rho_{\mathrm{DNR}} := \left| 1 - \theta \frac{\cosh(\alpha T) + \beta \sinh(\alpha T)}{\sinh(\alpha \Gamma) \big[\sinh(\alpha(T-\Gamma)) + \beta \cosh(\alpha(T-\Gamma)) \big]} \right|.
$$

Convergence tests (DNR)



Optimal relaxation parameter

- The Dirichlet-Neumann method converges if $\rho < 1$ with

$$\rho = \max_{d_i \in \Lambda(A)} \left| 1 - \theta \frac{\cosh(\alpha_i T) + \beta_i \sinh(\alpha_i T)}{\sinh(\alpha_i \Gamma)\left(\sinh(\alpha_i (T-\Gamma)) + \beta_i \cosh(\alpha_i (T-\Gamma))\right)} \right|.$$

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$$\lim_{d_i \to 0} \rho_{\mathrm{DNR}}(\theta^*) = \lim_{d_i \to \infty} \rho_{\mathrm{DNR}}(\theta^*),$$

i.e.,

$$\theta^* := \frac{2}{2 + \frac{\cosh\left(\frac{1}{\sqrt{\nu}} T\right) + \frac{\gamma}{\sqrt{\nu}} \sinh\left(\frac{1}{\sqrt{\nu}} T\right)}{\sinh(\frac{1}{\sqrt{\nu}}\Gamma)\left(\sinh\left(\frac{1}{\sqrt{\nu}}(T-\Gamma)\right) + \frac{\gamma}{\sqrt{\nu}} \cosh\left(\frac{1}{\sqrt{\nu}}(T-\Gamma)\right)\right)}}.$$

Neumann-Neumann (Bourgat, Glowinski, Tallec, Vidrascu 1989)

- $\Omega_1 := (0, \Gamma), \Omega_2 := (\Gamma, T)$ with Γ the interface. For $j = 1, 2$

$$
\begin{aligned}
\nu \ddot{e}_j^k - (\nu d_i^2 + 1) e_j^k &= 0, & \nu \ddot{\psi}_j^k - (\nu d_i^2 + 1) \psi_j^k &= 0, \\
e_1^k(0) &= 0, & \psi_1^k(0) &= 0, \\
\nu \dot{e}_2^k(T) + \nu d_i e_2^k(T) &= 0, & \psi_2^k(T) &= 0, \\
e_j^k(\Gamma) &= {\color{red} e_\Gamma^{k-1}}, & \partial_{n_j} \psi_j^k|_\Gamma &= \partial_{n_1} y_1^k|_\Gamma + \partial_{n_2} y_2^k|_\Gamma.
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- Solution:

$$
\begin{aligned}
e_1^k(t) =& e_\Gamma^{k-1} \frac{\sinh(\alpha t)}{\sinh(\alpha \Gamma)},\\
e_2^k(t) =& e_\Gamma^{k-1} \frac{\cosh(\alpha(T - t)) + \beta \sinh(\alpha(T - t))}{\cosh(\alpha(T - \Gamma)) + \beta \sinh(\alpha(T - \Gamma))}.
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\end{aligned}
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$$
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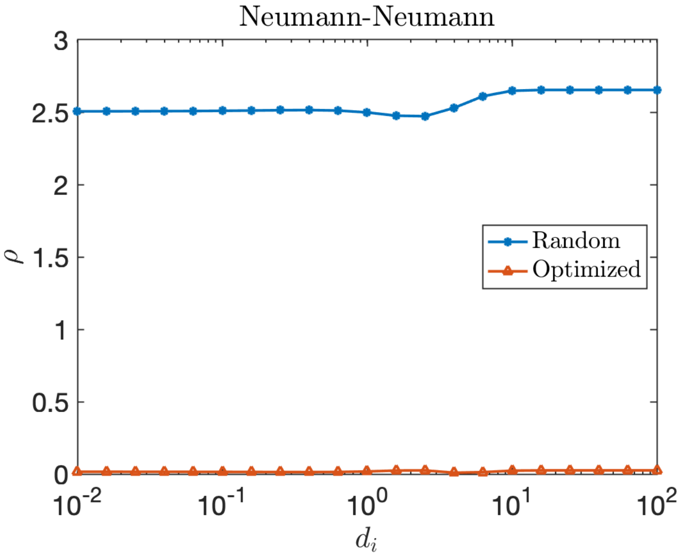
$$

Convergence tests (NN)



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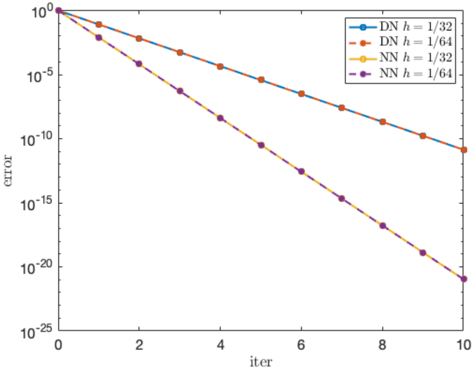
Numerical tests

- Domain: $(x, t) \in (0, 1) \times (0, 3)$, $\nu = 1$
- Discretization: $h = 1/32$ and $h = 1/64$ both in time and in space
- Two temporal subdomains: $\Omega_1 = (0, 1)$, $\Omega_2 = (1, 3)$
- Optimal θ: $\theta^*_{\mathrm{DN}} \approx 0.459$, $\theta^*_{\mathrm{NN}} \approx 0.252$



What else?

- Optimal control under H^{-1} regularization

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- Idea for parabolic case: $(-\partial_t - \Delta_x) \circ (T - \cdot)$?

Thanks for your attention !