Application of time decomposition to PDE-constrained optimization problems

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Matematikcentrums
Lunds Universitet

Fast solvers for nonlinear time-dependent problems

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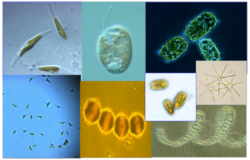
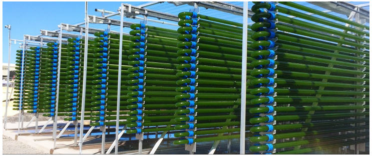
Application: microalgae production





Application: microalgae production

Photobioreactors:

- technology: chemostats, raceway, tubular, etc
- position: indoor / outdoor
- scale: lab / industrial
- purpose: wastewater treatment, biomass production, etc

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- position: indoor / outdoor
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Challenges:

- both physical and biological behavior
- coupled models
- different timescales
- macro / micro levels
- uncertainties: environment, measurement, etc

Example: optimize productivity in chemostats

Maximize biomass productivity $P(X)$

Governing equation: reaction-diffusion

$$\partial_t X - \kappa \Delta X + R(X) = f$$

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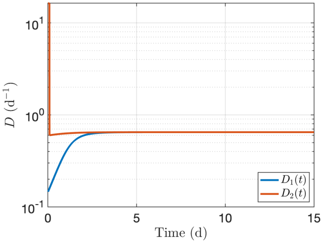
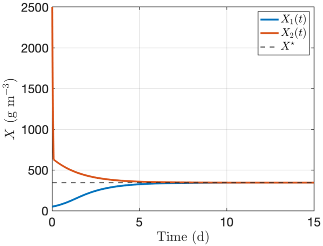
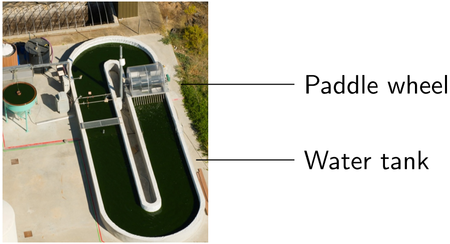
e.g., control with the dilution rate $R(X) = D(X)$





Example: optimize growth in raceway ponds



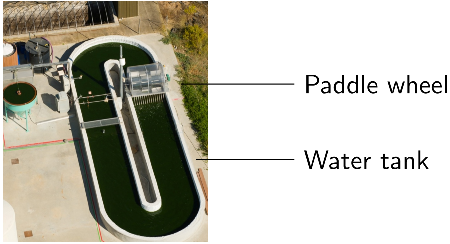
Example: optimize growth in raceway ponds

Maximize average growth rate $\bar{\mu}(X)$

Governing physics: hydrodynamics
    Navier-Stokes, Shallow water, etc

Microalgae movement: advection-diffusion

$$\partial_t X + \boldsymbol{u} \cdot \nabla X - \kappa \Delta X = f$$



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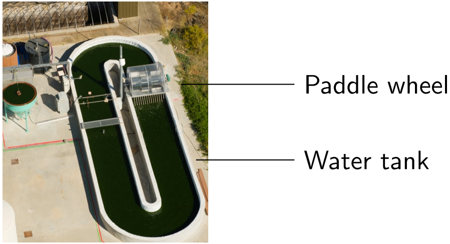

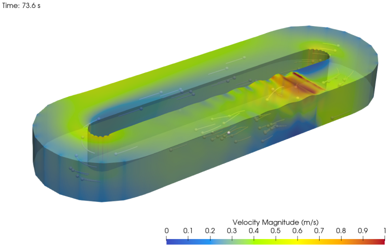

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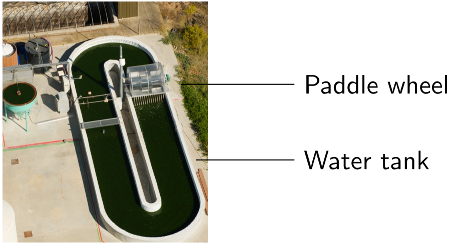
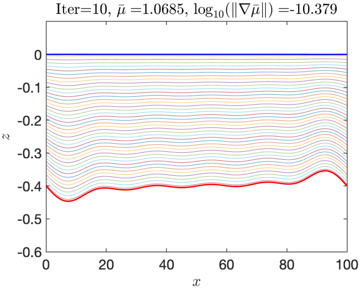
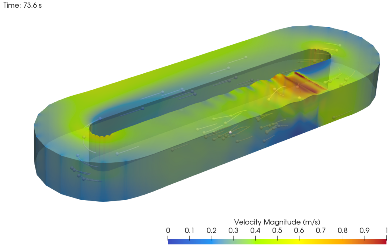
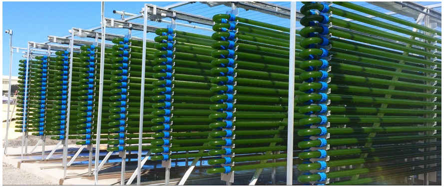
e.g., design best shape of the topography







Example: optimize temperature distribution in tubular bioreactor

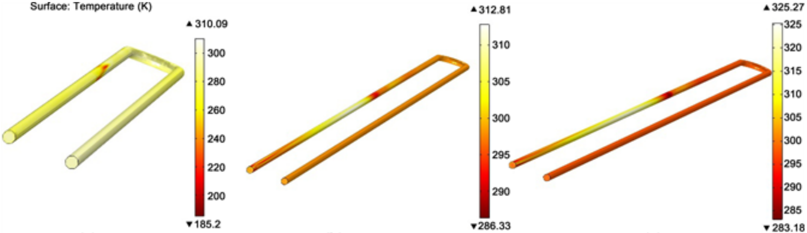


Minimize temperature distribution w.r.t. target $\hat{y}$

$$J(y,u) = \frac{1}{2}\|y-\hat{y}\|^2 + \frac{\nu}{2}\|u\|^2$$

Governing equation: heat equation

$$\partial_t y - \kappa \Delta y = u$$



(a) (b) (c)

PDE-constrained optimization problems

Optimize some functional of interest $J(y, u)$

Problem **constrained** by a PDE (or PDE-ODE coupled) system $F(y, u) = 0$

Find **optimal solutions** (y^*, u^*)

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Lagrange multiplier λ

$$\mathcal{L}(y, \lambda, u) = J(y, u) + \langle \lambda, F(y, u) \rangle$$

Derive first-order optimality system (necessary condition !)

$$\mathcal{L}_y(y^*, \lambda^*, u^*) = 0, \quad \mathcal{L}_\lambda(y^*, \lambda^*, u^*) = 0, \quad \mathcal{L}_u(y^*, \lambda^*, u^*) = 0$$

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Challenge: Solve can be time consuming, *e.g.*, complex systems, higher dimension

Goal: solve it in parallel using domain decomposition

Model problem

Temperature control w.r.t. a target:

minimize the functional $J(y,u) := \frac{1}{2}\|y-\hat{y}\|^2_{L^2(Q)} + \frac{\nu}{2}\|u\|^2_{L^2(Q)}$, $\hat{y} \in L^2(Q)$, $\nu > 0$

constrained by $\partial_t y - \Delta y = u$ in $Q := \Omega \times (0,T)$

completed with initial condition y_0 and boundary condition $y = 0$ on $\Sigma := \partial\Omega \times (0,T)$

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Derive first-order optimality system

$$\begin{aligned} \partial_t y - \Delta y &= u \quad \text{in } Q & \qquad \partial_t \lambda + \Delta\lambda &= y - \hat{y} \quad \text{in } Q \\ y &= 0 \quad \text{on } \Sigma & \lambda &= 0 \quad \text{on } \Sigma, \\ y &= y_0 \quad \text{on } \Sigma_0 & \lambda &= 0 \quad \text{on } \Sigma_T \end{aligned}$$

$$\nu u - \lambda = 0 \quad \text{in } Q$$

Model problem

Reduced optimality system (forward-backward)

$$\begin{aligned} \partial_t y - \Delta y &= \nu^{-1}\lambda && \text{in } Q \qquad & \partial_t \lambda + \Delta \lambda &= y - \hat{y} && \text{in } Q \\ y &= 0 && \text{on } \Sigma & \lambda &= 0 && \text{on } \Sigma \\ y &= y_0 && \text{on } \Sigma_0 & \lambda &= 0 && \text{on } \Sigma_T \end{aligned}$$

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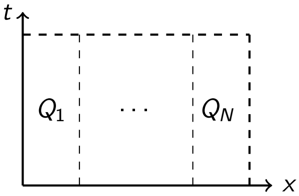
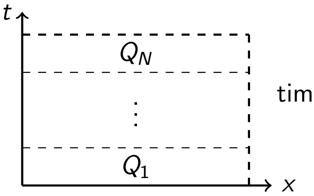
Apply domain decomposition methods

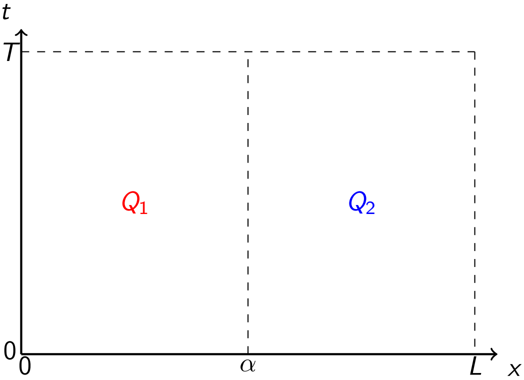
space decomposition





time decomposition

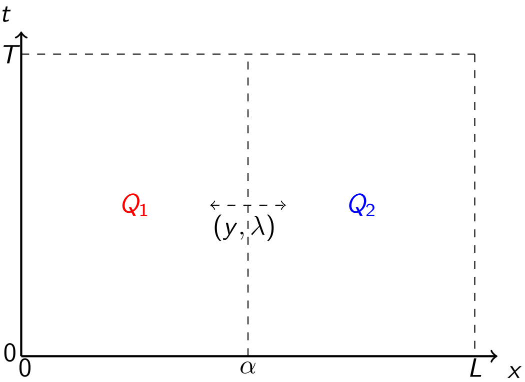
Space decomposition (Waveform method)



Subdomains: $Q_1 = (0,\alpha) \times (0,T)$ and $Q_2 = (\alpha, L) \times (0,T)$

$$\begin{aligned} \partial_t y - \partial_{xx} y &= \nu^{-1}\lambda & \qquad \partial_t \lambda + \partial_{xx}\lambda &= y - \hat{y} \\ y(0,t) &= 0 & \lambda(0,t) &= 0 \\ y(L,t) &= 0 & \lambda(L,t) &= 0 \\ y(x,0) &= y_0(x) & \lambda(x,T) &= 0 \end{aligned}$$

Space decomposition (Waveform method)



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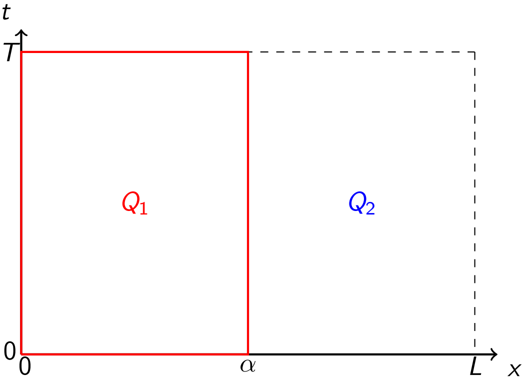
Alternating Schwarz waveform



In $Q_1 = (0,\alpha)\times(0,T)$

$$\partial_t y_1^\ell - \partial_{xx} y_1^\ell = \nu^{-1}\lambda_1^\ell \qquad \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell = y_1^\ell - \hat{y}_1$$

$$y_1^\ell(0,t) = 0 \qquad \lambda_1^\ell(0,t) = 0$$

$$y_1^\ell(x,0) = y_{1,0}(x) \qquad \lambda_1^\ell(x,T) = 0$$

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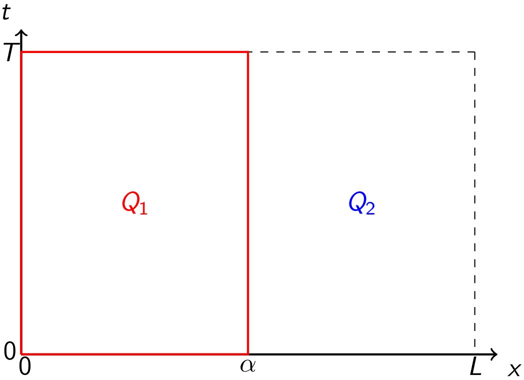

In $Q_1 = (0, \alpha) \times (0, T)$

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\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell & \qquad \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0,t) &= 0 & \lambda_1^\ell(0,t) &= 0 \\
y_1^\ell(\alpha,t) &= y_2^{\ell-1}(\alpha,t) & \lambda_1^\ell(\alpha,t) &= \lambda_2^{\ell-1}(\alpha,t) \\
y_1^\ell(x,0) &= y_{1,0}(x) & \lambda_1^\ell(x,T) &= 0
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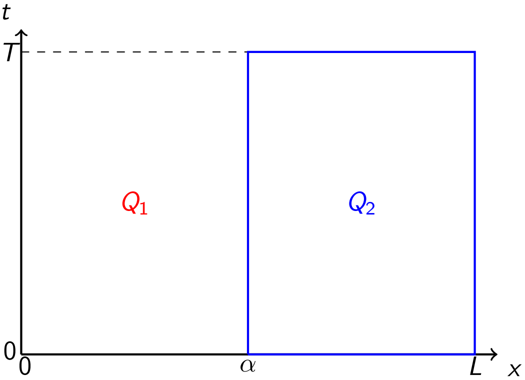

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\end{aligned}$$

In $Q_2 = (\alpha,L)\times(0,T)$

$$\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1}\lambda_2^\ell & \qquad \partial_t \lambda_2^\ell + \partial_{xx}\lambda_2^\ell &= y_2^\ell - \hat{y}_2\\
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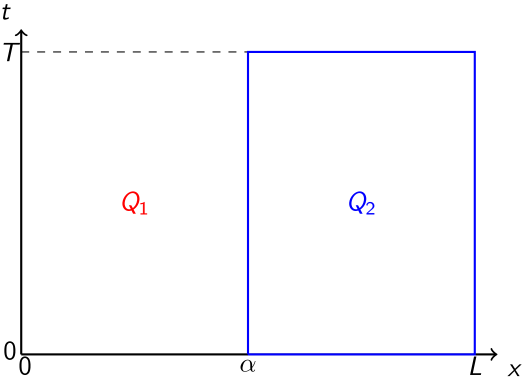
Alternating Schwarz waveform



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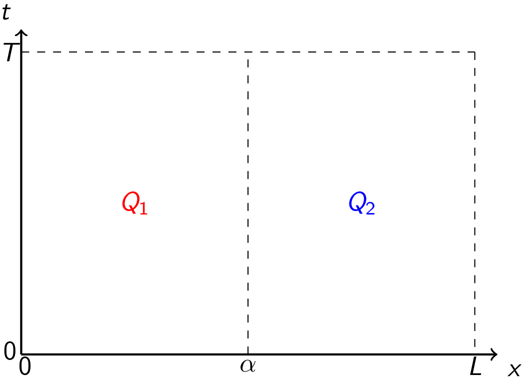
\end{aligned}$$

Parallel Schwarz waveform



In $Q_1 = (0,\alpha)\times(0,T)$

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\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell & \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell &= y_1^\ell - \hat{y}_1\\
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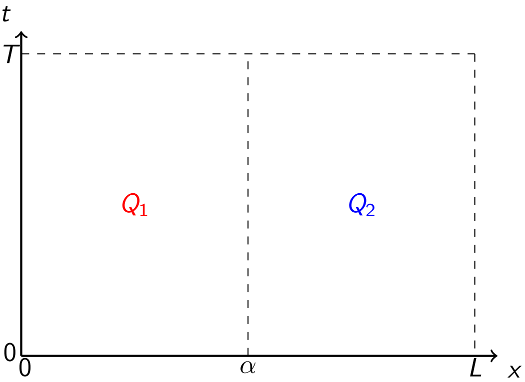
Convergence of alternating Schwarz waveform



In $Q_1 = (0,\alpha) \times (0,T)$

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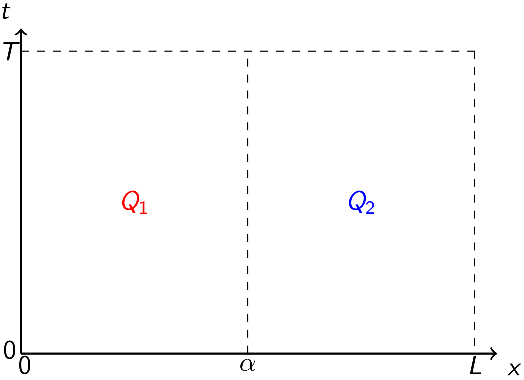

The algorithm **does not converge** without overlap !

In $Q_1 = (0,\alpha) \times (0,T)$

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell & \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0,t) &= 0 & \lambda_1^\ell(0,t) &= 0 \\ y_1^\ell(\alpha,t) &= y_2^{\ell-1}(\alpha,t) & \lambda_1^\ell(\alpha,t) &= \lambda_2^{\ell-1}(\alpha,t) \\ y_1^\ell(x,0) &= y_{1,0}(x) & \lambda_1^\ell(x,T) &= 0 \end{aligned}$$

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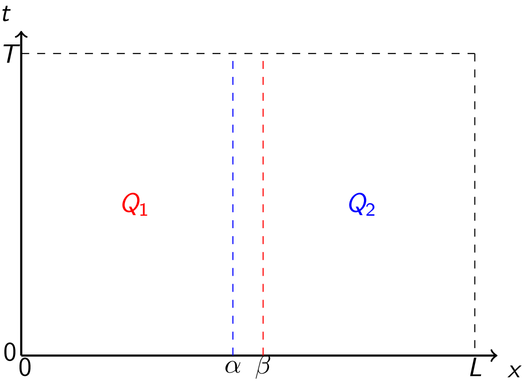

In $Q_1 = (0, \beta) \times (0, T)$

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} \lambda_1^\ell & \partial_t \lambda_1^\ell + \partial_{xx} \lambda_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0,t) &= 0 & \lambda_1^\ell(0,t) &= 0 \\ y_1^\ell(\beta,t) &= y_2^{\ell-1}(\beta,t) & \lambda_1^\ell(\beta,t) &= \lambda_2^{\ell-1}(\beta,t) \\ y_1^\ell(x,0) &= y_{1,0}(x) & \lambda_1^\ell(x,T) &= 0 \end{aligned}$$

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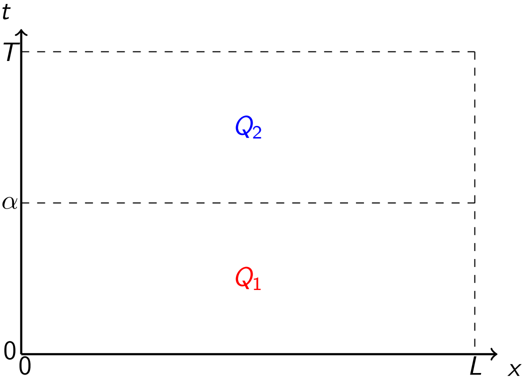
The algorithm now **converges** with overlap !

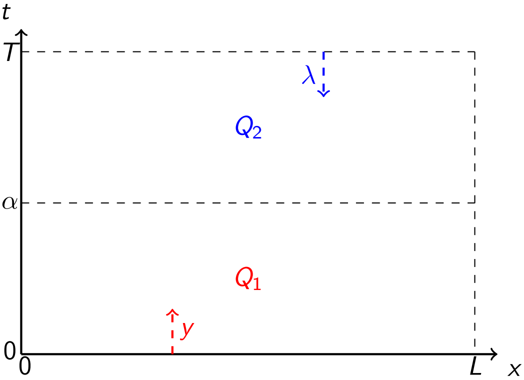
Time decomposition



Subdomains: $Q_1 = (0, L) \times (0, \alpha)$ and $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y - \partial_{xx} y &= \nu^{-1}\lambda & \qquad \partial_t \lambda + \partial_{xx}\lambda &= y - \hat{y} \\ y(0, t) &= 0 & \lambda(0, t) &= 0 \\ y(L, t) &= 0 & \lambda(L, t) &= 0 \\ y(x, 0) &= y_0(x) & \lambda(x, T) &= 0 \end{aligned}$$

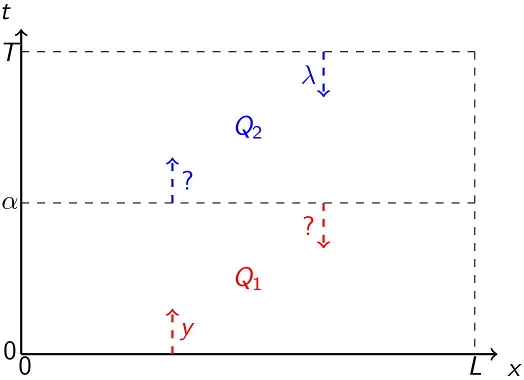
Time decomposition



Subdomains: $Q_1 = (0, L) \times (0, \alpha)$ and $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y - \partial_{xx} y &= \nu^{-1}\lambda & \qquad \partial_t \lambda + \partial_{xx}\lambda &= y - \hat{y} \\ y(0,t) &= 0 & \lambda(0,t) &= 0 \\ y(L,t) &= 0 & \lambda(L,t) &= 0 \\ y(x,0) &= y_0(x) & \lambda(x,T) &= 0 \end{aligned}$$

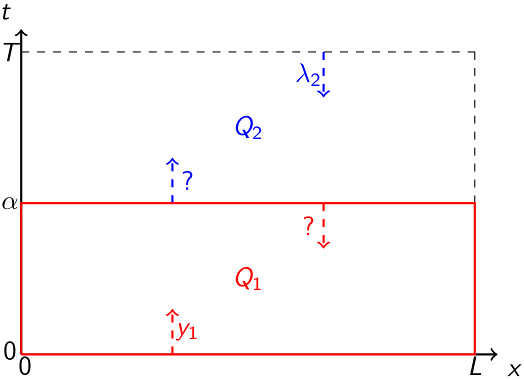
Time decomposition



Subdomains: $Q_1 = (0,L) \times (0,\alpha)$ and $Q_2 = (0,L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y - \partial_{xx} y &= \nu^{-1}\lambda & \qquad \partial_t \lambda + \partial_{xx}\lambda &= y - \hat{y} \\ y(0,t) &= 0 & \lambda(0,t) &= 0 \\ y(L,t) &= 0 & \lambda(L,t) &= 0 \\ y(x,0) &= y_0(x) & \lambda(x,T) &= 0 \end{aligned}$$

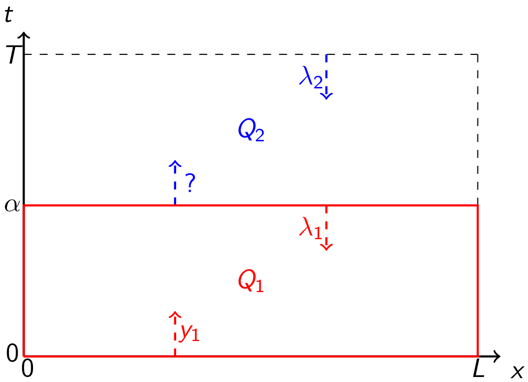
Alternating Schwarz in time



In $Q_1 = (0, L) \times (0, \alpha)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} \lambda_1^\ell & \partial_t \lambda_1^\ell + \partial_{xx} \lambda_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) &= 0 & \lambda_1^\ell(0, t) &= 0 \\
y_1^\ell(L, t) &= 0 & \lambda_1^\ell(L, t) &= 0 \\
y_1^\ell(x, 0) &= y_0(x)
\end{aligned}
$$

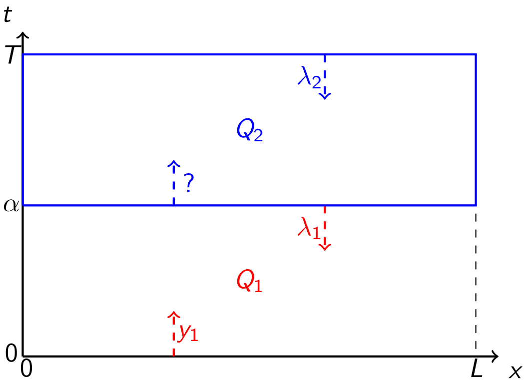
Alternating Schwarz in time



In $Q_1 = (0,L)\times(0,\alpha)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell & \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0,t) &= 0 & \lambda_1^\ell(0,t) &= 0 \\
y_1^\ell(L,t) &= 0 & \lambda_1^\ell(L,t) &= 0 \\
y_1^\ell(x,0) &= y_0(x) & \lambda_1^\ell(x,\alpha) &= \lambda_2^{\ell-1}(x,\alpha)
\end{aligned}
$$

Alternating Schwarz in time



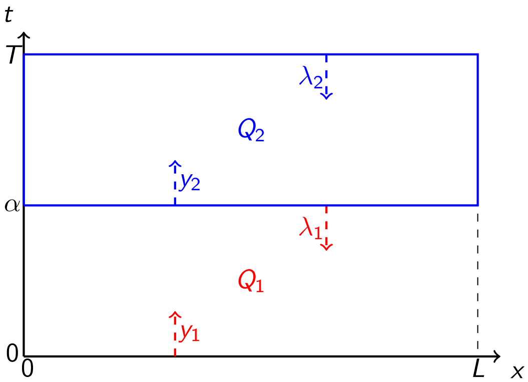
In $Q_1 = (0, L) \times (0, \alpha)$

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell & \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0,t) &= 0 & \lambda_1^\ell(0,t) &= 0 \\ y_1^\ell(L,t) &= 0 & \lambda_1^\ell(L,t) &= 0 \\ y_1^\ell(x,0) &= y_0(x) & \lambda_1^\ell(x,\alpha) &= \lambda_2^{\ell-1}(x,\alpha) \end{aligned}$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1}\lambda_2^\ell & \partial_t \lambda_2^\ell + \partial_{xx}\lambda_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0,t) &= 0 & \lambda_2^\ell(0,t) &= 0 \\ y_2^\ell(L,t) &= 0 & \lambda_2^\ell(L,t) &= 0 \\ & & \lambda_2^\ell(x,T) &= 0 \end{aligned}$$

Alternating Schwarz in time



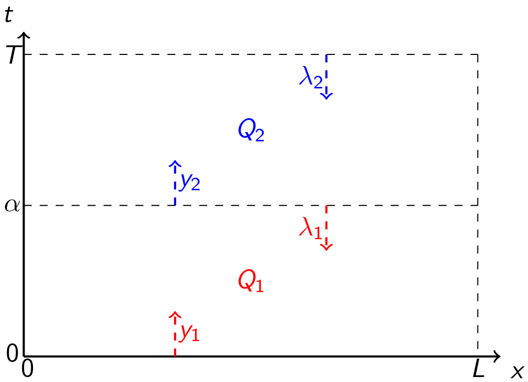
In $Q_1 = (0, L) \times (0, \alpha)$

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell & \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0,t) &= 0 & \lambda_1^\ell(0,t) &= 0 \\ y_1^\ell(L,t) &= 0 & \lambda_1^\ell(L,t) &= 0 \\ y_1^\ell(x,0) &= y_0(x) & \lambda_1^\ell(x,\alpha) &= \lambda_2^{\ell-1}(x,\alpha) \end{aligned}$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1}\lambda_2^\ell & \partial_t \lambda_2^\ell + \partial_{xx}\lambda_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0,t) &= 0 & \lambda_2^\ell(0,t) &= 0 \\ y_2^\ell(L,t) &= 0 & \lambda_2^\ell(L,t) &= 0 \\ y_2^\ell(x,\alpha) &= y_1^\ell(x,\alpha) & \lambda_2^\ell(x,T) &= 0 \end{aligned}$$

Parallel Schwarz in time



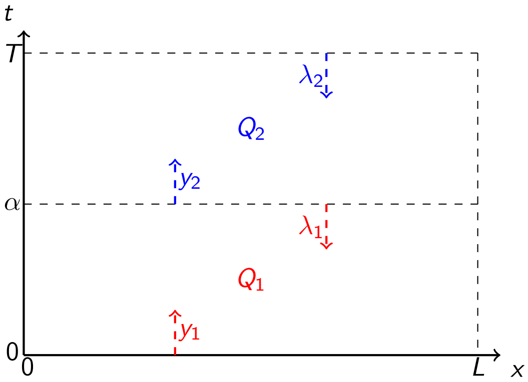
In $Q_1 = (0, L) \times (0, \alpha)$

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell & \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0,t) &= 0 & \lambda_1^\ell(0,t) &= 0 \\ y_1^\ell(L,t) &= 0 & \lambda_1^\ell(L,t) &= 0 \\ y_1^\ell(x,0) &= y_0(x) & \lambda_1^\ell(x,\alpha) &= \lambda_2^{\ell-1}(x,\alpha) \end{aligned}$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1}\lambda_2^\ell & \partial_t \lambda_2^\ell + \partial_{xx}\lambda_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0,t) &= 0 & \lambda_2^\ell(0,t) &= 0 \\ y_2^\ell(L,t) &= 0 & \lambda_2^\ell(L,t) &= 0 \\ y_2^\ell(x,\alpha) &= y_1^{\ell-1}(x,\alpha) & \lambda_2^\ell(x,T) &= 0 \end{aligned}$$

Convergence of alternating Schwarz in time



How about the convergence ?

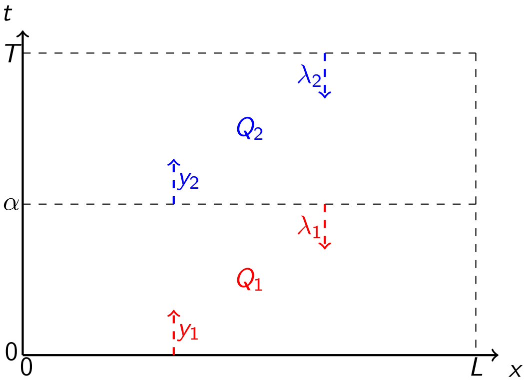
In $Q_1 = (0, L) \times (0, \alpha)$

$$\begin{array}{cc}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell = \nu^{-1}\lambda_1^\ell & \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell = y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) = 0 & \lambda_1^\ell(0, t) = 0 \\
y_1^\ell(L, t) = 0 & \lambda_1^\ell(L, t) = 0 \\
y_1^\ell(x, 0) = y_0(x) & \lambda_1^\ell(x, \alpha) = \lambda_2^{\ell-1}(x, \alpha)
\end{array}$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{array}{cc}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell = \nu^{-1}\lambda_2^\ell & \partial_t \lambda_2^\ell + \partial_{xx}\lambda_2^\ell = y_2^\ell - \hat{y}_2 \\
y_2^\ell(0, t) = 0 & \lambda_2^\ell(0, t) = 0 \\
y_2^\ell(L, t) = 0 & \lambda_2^\ell(L, t) = 0 \\
y_2^\ell(x, \alpha) = y_1^\ell(x, \alpha) & \lambda_2^\ell(x, T) = 0
\end{array}$$

Convergence of alternating Schwarz in time



How about the convergence ?
It **does converge** without overlap !

In $Q_1 = (0, L) \times (0, \alpha)$

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell & \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0,t) &= 0 & \lambda_1^\ell(0,t) &= 0 \\ y_1^\ell(L,t) &= 0 & \lambda_1^\ell(L,t) &= 0 \\ y_1^\ell(x,0) &= y_0(x) & \lambda_1^\ell(x,\alpha) &= \lambda_2^{\ell-1}(x,\alpha) \end{aligned}$$

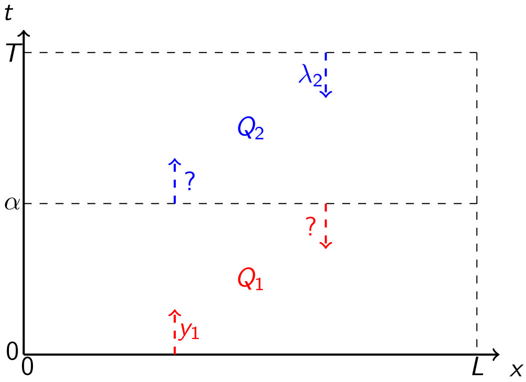
In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1}\lambda_2^\ell & \partial_t \lambda_2^\ell + \partial_{xx}\lambda_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0,t) &= 0 & \lambda_2^\ell(0,t) &= 0 \\ y_2^\ell(L,t) &= 0 & \lambda_2^\ell(L,t) &= 0 \\ y_2^\ell(x,\alpha) &= y_1^\ell(x,\alpha) & \lambda_2^\ell(x,T) &= 0 \end{aligned}$$

Intuition

Relations

$$\lambda_j^\ell = \nu(\partial_t y_j^\ell - \partial_{xx} y_j^\ell)$$
$$y_j^\ell = \partial_t \lambda_j^\ell + \partial_{xx} \lambda_j^\ell + \hat{y}_j$$



In $Q_1 = (0,L)\times(0,\alpha)$

$$\partial_t y_1^\ell - \partial_{xx} y_1^\ell = \nu^{-1}\lambda_1^\ell \qquad \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell = y_1^\ell - \hat{y}_1$$
$$y_1^\ell(0,t) = 0 \qquad \lambda_1^\ell(0,t) = 0$$
$$y_1^\ell(L,t) = 0 \qquad \lambda_1^\ell(L,t) = 0$$
$$y_1^\ell(x,0) = y_0(x)$$

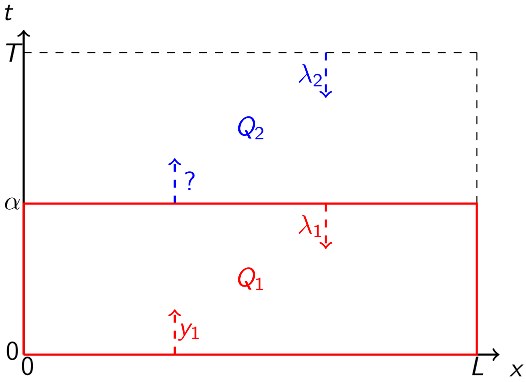
In $Q_2 = (0,L)\times(\alpha,T)$

$$\partial_t y_2^\ell - \partial_{xx} y_2^\ell = \nu^{-1}\lambda_2^\ell \qquad \partial_t \lambda_2^\ell + \partial_{xx}\lambda_2^\ell = y_2^\ell - \hat{y}_2$$
$$y_2^\ell(0,t) = 0 \qquad \lambda_2^\ell(0,t) = 0$$
$$y_2^\ell(L,t) = 0 \qquad \lambda_2^\ell(L,t) = 0$$
$$\lambda_2^\ell(x,T) = 0$$

Intuition

Relations

$$\lambda_j^\ell = \nu(\partial_t y_j^\ell - \partial_{xx} y_j^\ell)$$



In $Q_1 = (0, L) \times (0, \alpha)$

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell & \partial_t \lambda_1^\ell + \partial_{xx} \lambda_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0,t) &= 0 & \lambda_1^\ell(0,t) &= 0 \\ y_1^\ell(L,t) &= 0 & \lambda_1^\ell(L,t) &= 0 \\ y_1^\ell(x,0) &= y_0(x) & \lambda_1^\ell(x,\alpha) &= \lambda_2^{\ell-1}(x,\alpha) \end{aligned}$$

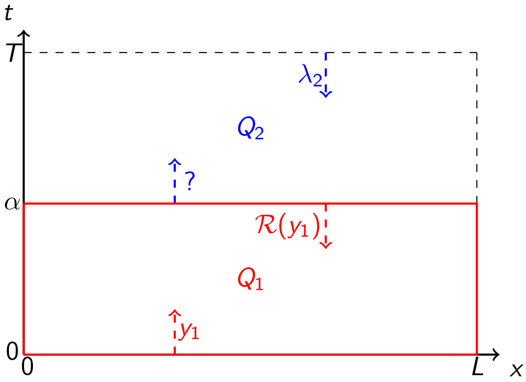
In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1}\lambda_2^\ell & \partial_t \lambda_2^\ell + \partial_{xx} \lambda_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0,t) &= 0 & \lambda_2^\ell(0,t) &= 0 \\ y_2^\ell(L,t) &= 0 & \lambda_2^\ell(L,t) &= 0 \\ & & \lambda_2^\ell(x,T) &= 0 \end{aligned}$$

Intuition

Relations

$$\lambda_j^\ell = \nu(\partial_t y_j^\ell - \partial_{xx} y_j^\ell)$$



In $Q_1 = (0, L) \times (0, \alpha)$

$$\partial_t y_1^\ell - \partial_{xx} y_1^\ell = \nu^{-1}\lambda_1^\ell \qquad \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell = y_1^\ell - \hat{y}_1$$

$$y_1^\ell(0,t) = 0 \qquad \lambda_1^\ell(0,t) = 0$$

$$y_1^\ell(L,t) = 0 \qquad \lambda_1^\ell(L,t) = 0$$

$$y_1^\ell(x,0) = y_0(x)$$

$$(\partial_t - \partial_{xx}) y_1^\ell(x,\alpha) = (\partial_t - \partial_{xx}) y_2^{\ell-1}(x,\alpha)$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\partial_t y_2^\ell - \partial_{xx} y_2^\ell = \nu^{-1}\lambda_2^\ell \qquad \partial_t \lambda_2^\ell + \partial_{xx}\lambda_2^\ell = y_2^\ell - \hat{y}_2$$

$$y_2^\ell(0,t) = 0 \qquad \lambda_2^\ell(0,t) = 0$$

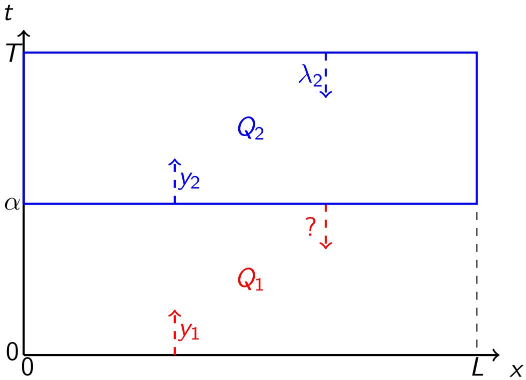
$$y_2^\ell(L,t) = 0 \qquad \lambda_2^\ell(L,t) = 0$$

$$\lambda_2^\ell(x,T) = 0$$

Intuition

Relations

$$y_j^\ell = \partial_t \lambda_j^\ell + \partial_{xx} \lambda_j^\ell + \hat{y}_j$$



In $Q_1 = (0, L) \times (0, \alpha)$

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} \lambda_1^\ell & \partial_t \lambda_1^\ell + \partial_{xx} \lambda_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0, t) &= 0 & \lambda_1^\ell(0, t) &= 0 \\ y_1^\ell(L, t) &= 0 & \lambda_1^\ell(L, t) &= 0 \\ y_1^\ell(x, 0) &= y_0(x) \end{aligned}$$

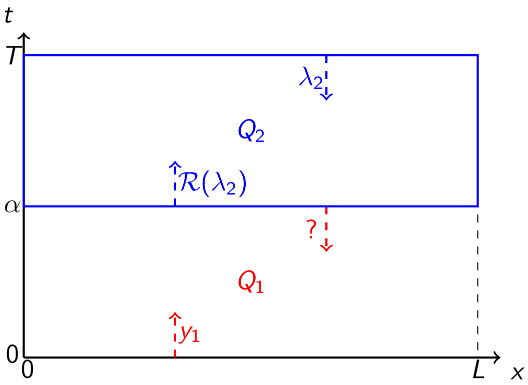
In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} \lambda_2^\ell & \partial_t \lambda_2^\ell + \partial_{xx} \lambda_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0, t) &= 0 & \lambda_2^\ell(0, t) &= 0 \\ y_2^\ell(L, t) &= 0 & \lambda_2^\ell(L, t) &= 0 \\ y_2^\ell(x, \alpha) &= y_1^\ell(x, \alpha) & \lambda_2^\ell(x, T) &= 0 \end{aligned}$$

Intuition

Relations

$$y_j^\ell = \partial_t \lambda_j^\ell + \partial_{xx}\lambda_j^\ell + \hat{y}_j$$



In $Q_1 = (0, L) \times (0, \alpha)$

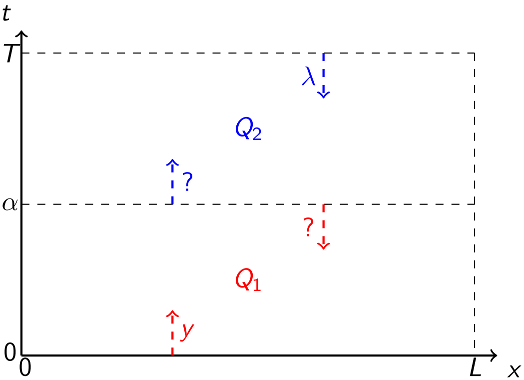
$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell & \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0,t) &= 0 & \lambda_1^\ell(0,t) &= 0 \\ y_1^\ell(L,t) &= 0 & \lambda_1^\ell(L,t) &= 0 \\ y_1^\ell(x,0) &= y_0(x) \end{aligned}$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1}\lambda_2^\ell & \partial_t \lambda_2^\ell + \partial_{xx}\lambda_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0,t) &= 0 & \lambda_2^\ell(0,t) &= 0 \\ y_2^\ell(L,t) &= 0 & \lambda_2^\ell(L,t) &= 0 \\ & & \lambda_2^\ell(x,T) &= 0 \end{aligned}$$

$$(\partial_t + \partial_{xx})\lambda_2^\ell(x,\alpha) = (\partial_t + \partial_{xx})\lambda_1^\ell(x,\alpha)$$

Observation



- **Dirichlet** condition transform to **Robin** type condition !
- Forward-backward **might be** less important ?

In $Q_1 = (0, L) \times (0, \alpha)$

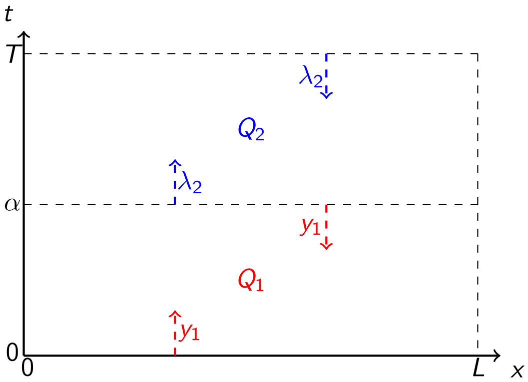
$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell & \partial_t \lambda_1^\ell + \partial_{xx} \lambda_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0,t) &= 0 & \lambda_1^\ell(0,t) &= 0 \\ y_1^\ell(L,t) &= 0 & \lambda_1^\ell(L,t) &= 0 \\ y_1^\ell(x,0) &= y_0(x) & & \end{aligned}$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1}\lambda_2^\ell & \partial_t \lambda_2^\ell + \partial_{xx} \lambda_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0,t) &= 0 & \lambda_2^\ell(0,t) &= 0 \\ y_2^\ell(L,t) &= 0 & \lambda_2^\ell(L,t) &= 0 \\ & & \lambda_2^\ell(x,T) &= 0 \end{aligned}$$

Variants of alternating Schwarz in time

Exchange λ and y



In $Q_1 = (0, L) \times (0, \alpha)$

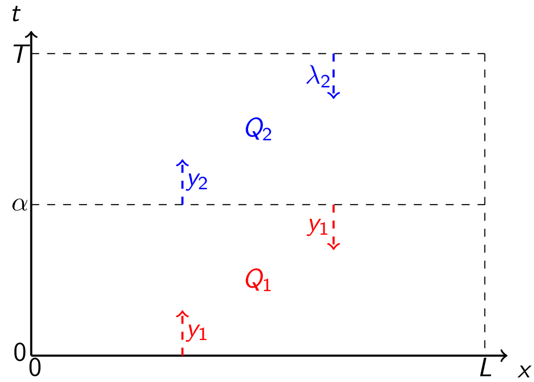
$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell & \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0,t) &= 0 & \lambda_1^\ell(0,t) &= 0 \\ y_1^\ell(L,t) &= 0 & \lambda_1^\ell(L,t) &= 0 \\ y_1^\ell(x,0) &= y_0(x) & y_1^\ell(x,\alpha) &= y_2^{\ell-1}(x,\alpha) \end{aligned}$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1}\lambda_2^\ell & \partial_t \lambda_2^\ell + \partial_{xx}\lambda_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0,t) &= 0 & \lambda_2^\ell(0,t) &= 0 \\ y_2^\ell(L,t) &= 0 & \lambda_2^\ell(L,t) &= 0 \\ \lambda_2^\ell(x,\alpha) &= \lambda_1^\ell(x,\alpha) & \lambda_2^\ell(x,T) &= 0 \end{aligned}$$

Variants of alternating Schwarz in time

Only with y



In $Q_1 = (0, L) \times (0, \alpha)$

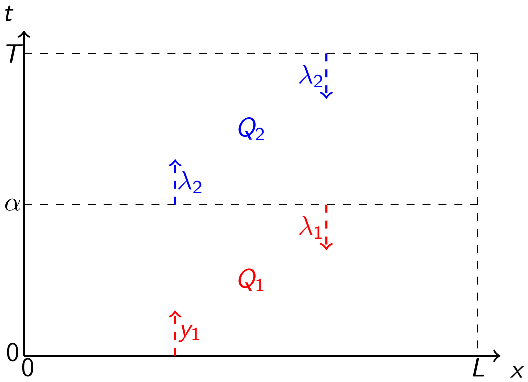
$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell & \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0,t) &= 0 & \lambda_1^\ell(0,t) &= 0 \\ y_1^\ell(L,t) &= 0 & \lambda_1^\ell(L,t) &= 0 \\ y_1^\ell(x,0) &= y_0(x) & y_1^\ell(x,\alpha) &= y_2^{\ell-1}(x,\alpha) \end{aligned}$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1}\lambda_2^\ell & \partial_t \lambda_2^\ell + \partial_{xx}\lambda_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0,t) &= 0 & \lambda_2^\ell(0,t) &= 0 \\ y_2^\ell(L,t) &= 0 & \lambda_2^\ell(L,t) &= 0 \\ y_2^\ell(x,\alpha) &= y_1^\ell(x,\alpha) & \lambda_2^\ell(x,T) &= 0x \end{aligned}$$

Variants of alternating Schwarz in time

Only with λ



In $Q_1 = (0, L) \times (0, \alpha)$

$$\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell & \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0,t) &= 0 & \lambda_1^\ell(0,t) &= 0 \\
y_1^\ell(L,t) &= 0 & \lambda_1^\ell(L,t) &= 0 \\
y_1^\ell(x,0) &= y_0(x) & \lambda_1^\ell(x,\alpha) &= \lambda_2^{\ell-1}(x,\alpha)
\end{aligned}$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1}\lambda_2^\ell & \partial_t \lambda_2^\ell + \partial_{xx}\lambda_2^\ell &= y_2^\ell - \hat{y}_2 \\
y_2^\ell(0,t) &= 0 & \lambda_2^\ell(0,t) &= 0 \\
y_2^\ell(L,t) &= 0 & \lambda_2^\ell(L,t) &= 0 \\
\lambda_2^\ell(x,\alpha) &= \lambda_1^\ell(x,\alpha) & \lambda_2^\ell(x,T) &= 0
\end{aligned}$$

Convergence of Dirichlet variants

Dirichlet type transmission condition:

name	SD_1	SD_2	SD_3	SD_4
Q_1	λ	y	y	λ
Q_2	y	λ	y	λ

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Analysis: semi-discretization in space

$$\begin{aligned}
\rho_{\mathrm{SD}_1} &= \max_{d_i \in D} \left| \frac{1+\gamma(\sigma_i \coth(b_i) - d_i)}{\nu(\sigma_i \coth(a_i) + d_i)(\sigma_i \coth(b_i) + d_i + \gamma\nu^{-1})} \right| \\
\rho_{\mathrm{SD}_2} &= \max_{d_i \in D} \left| \frac{\nu(\sigma_i \coth(a_i) + d_i)(\sigma_i \coth(b_i) + d_i + \gamma\nu^{-1})}{1+\gamma(\sigma_i \coth(b_i) - d_i)} \right| \\
\rho_{\mathrm{SD}_3} &= 1 \\
\rho_{\mathrm{SD}_4} &= 1
\end{aligned}$$

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$$\rho_{\text{SD}_2} = \max_{d_i \in D} \left| \frac{\nu(\sigma_i \coth(a_i) + d_i)(\sigma_i \coth(b_i) + d_i + \gamma\nu^{-1})}{1+\gamma(\sigma_i \coth(b_i) - d_i)} \right|$$

$$\rho_{\text{SD}_3} = 1$$

$$\rho_{\text{SD}_4} = 1$$

Relaxation: *e.g.*, $\lambda_1^\ell(x,\alpha) = \lambda_2^{\ell-1}(x,\alpha)$ to $\lambda_1^\ell(x,\alpha) = f^\ell(x)$ with

$$f^{\ell+1}(x) = (1-\theta)f^\ell(x) + \theta\lambda_2^\ell(x,\alpha) \qquad \theta \in (0,1)$$

How about Neumann condition ?

Neumann type transmission condition:

name	SN_1	SN_2	SN_3	SN_4
Q_1	$\partial_t \lambda$	$\partial_t y$	$\partial_t y$	$\partial_t \lambda$
Q_2	$\partial_t y$	$\partial_t \lambda$	$\partial_t y$	$\partial_t \lambda$

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Neumann type transmission condition:

name	SN_1	SN_2	SN_3	SN_4
Q_1	$\partial_t \lambda$	$\partial_t y$	$\partial_t y$	$\partial_t \lambda$
Q_2	$\partial_t y$	$\partial_t \lambda$	$\partial_t y$	$\partial_t \lambda$

Analysis: semi-discretization in space:

$$\begin{aligned}
\rho_{\text{SN}_1} &= \max_{d_i \in D} \left| \frac{1+\gamma(\sigma_i \tanh(b_i) - d_i)}{\nu(\sigma_i \tanh(a_i) + d_i)(\sigma_i \tanh(b_i) + d_i + \gamma\nu^{-1})} \right| \\
\rho_{\text{SN}_2} &= \max_{d_i \in D} \left| \frac{\nu(\sigma_i \tanh(a_i) + d_i)(\sigma_i \tanh(b_i) + d_i + \gamma\nu^{-1})}{1+\gamma(\sigma_i \tanh(b_i) - d_i)} \right| \\
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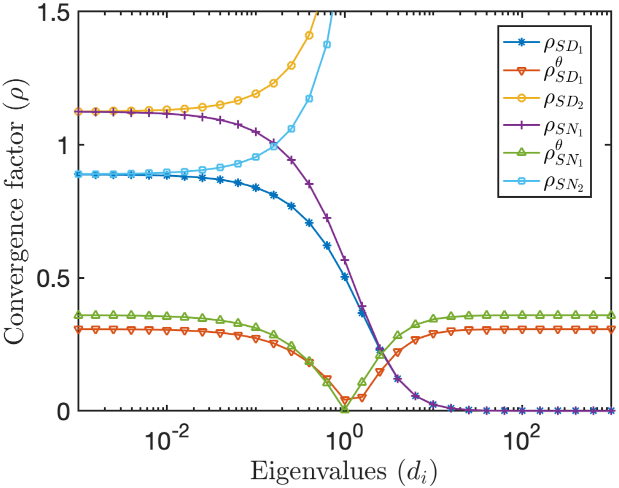
\end{aligned}$$

Relaxation: *e.g.*, $\partial_t \lambda_1^\ell(x,\alpha) = \partial_t \lambda_2^{\ell-1}(x,\alpha)$ to $\partial_t \lambda_1^\ell(x,\alpha) = f^\ell(x)$ with

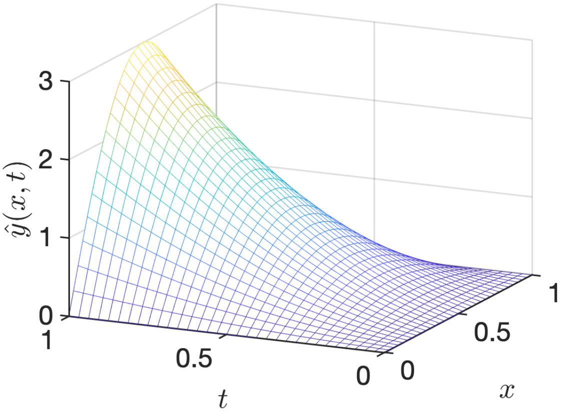
$$f^{\ell+1}(x) = (1-\theta) f^\ell(x) + \theta \partial_t \lambda_2^\ell(x,\alpha) \qquad \theta \in (0,1)$$

Convergence factor



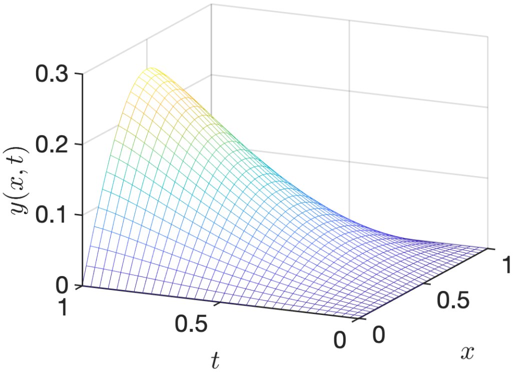
Numerical experiments

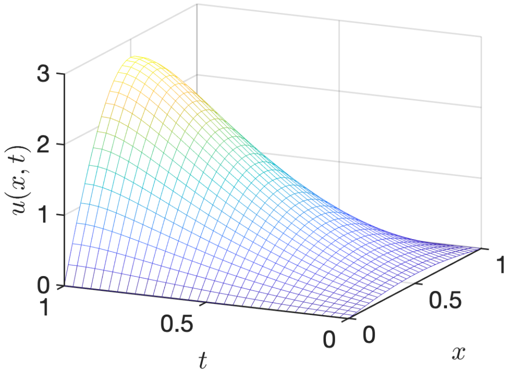
Test case: consider the target function $\hat{y}(x,t) = \sin(\pi x)(2t^2 + 2)$



Numerical experiments

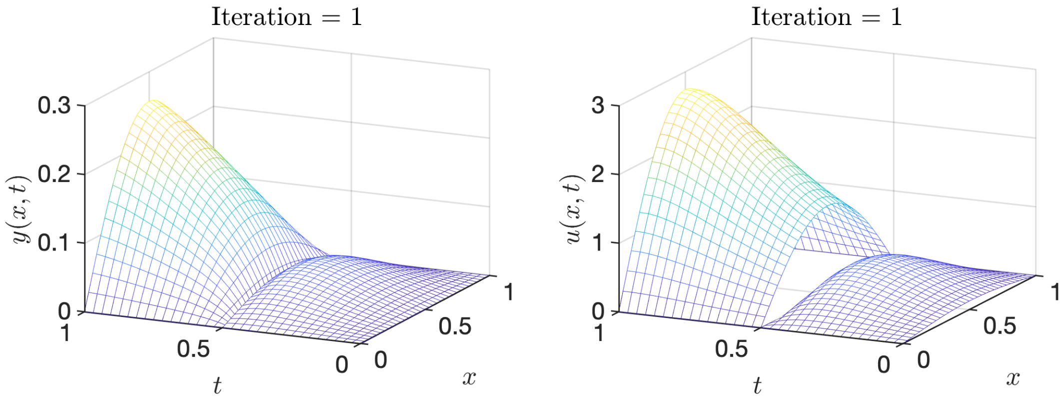
Numerical solutions: Crank-Nicolson with mesh size $h_t = h_x = \frac{1}{32}$ and penalization parameters: $\nu = 0.1$, $\gamma = 0.1$





Numerical experiments

Two subdomains:



Numerical experiments

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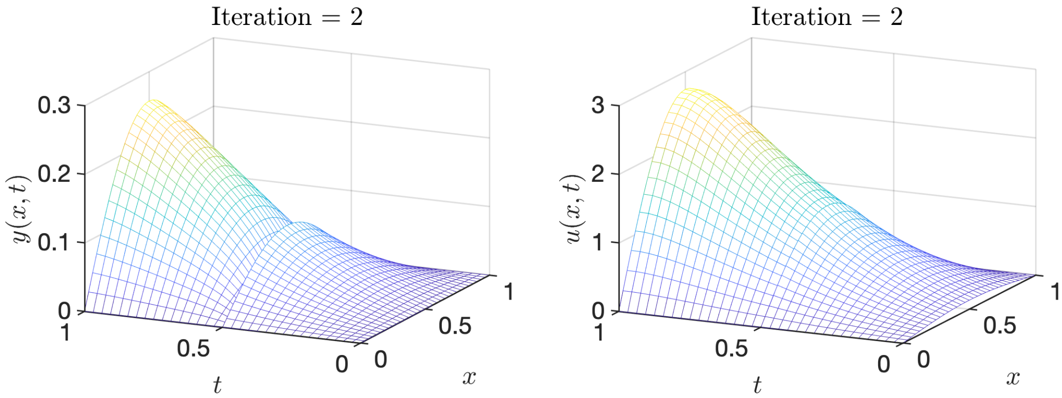

Numerical experiments

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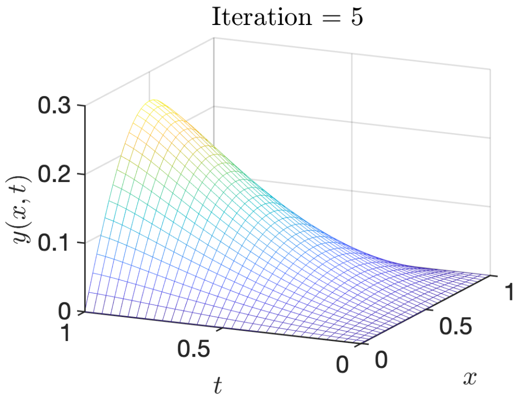
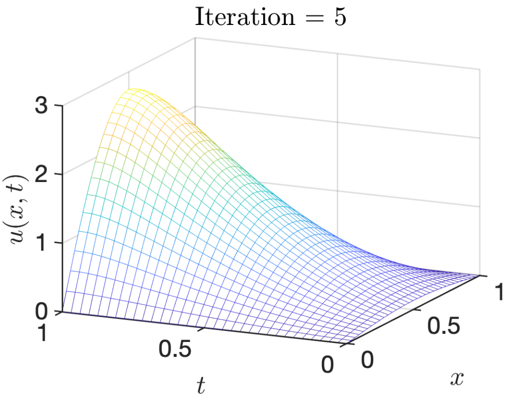
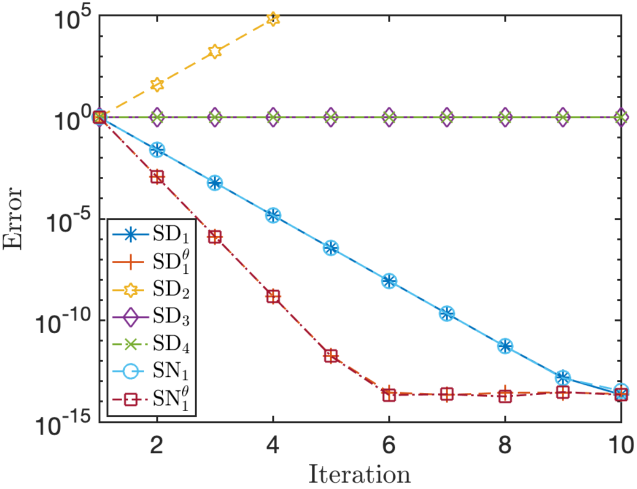
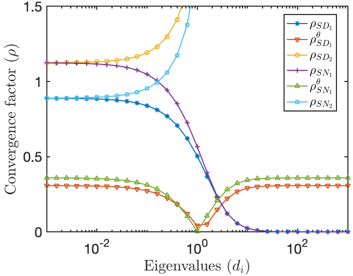




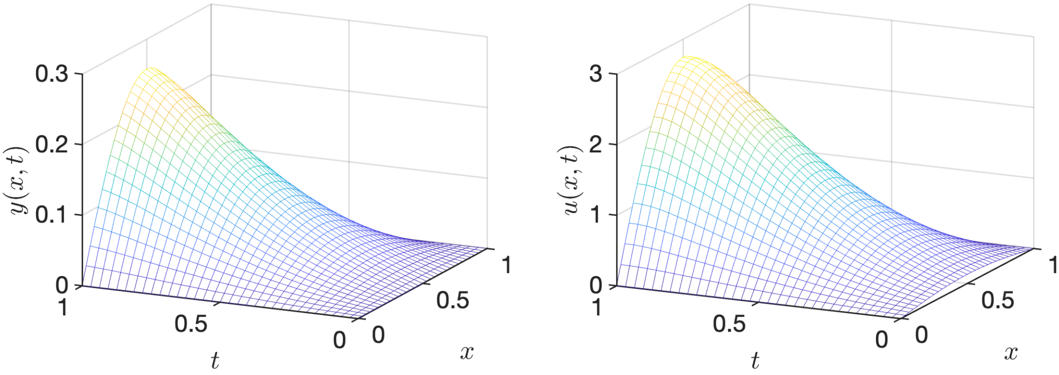
Convergence behavior





Weak scalability test

Numerical solutions: Crank-Nicolson with mesh size $h_t = h_x = \frac{1}{32}$ and penalization parameters: $\nu = 0.1$, $\gamma = 0.1$



Weak scalability test

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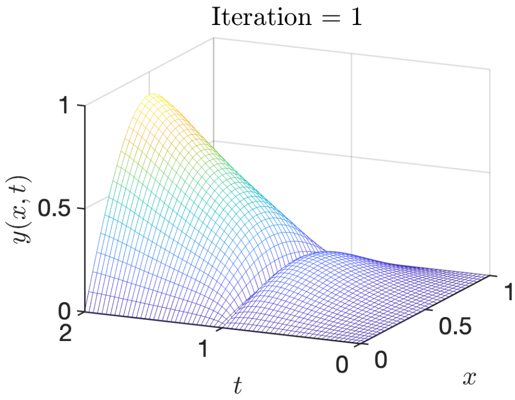

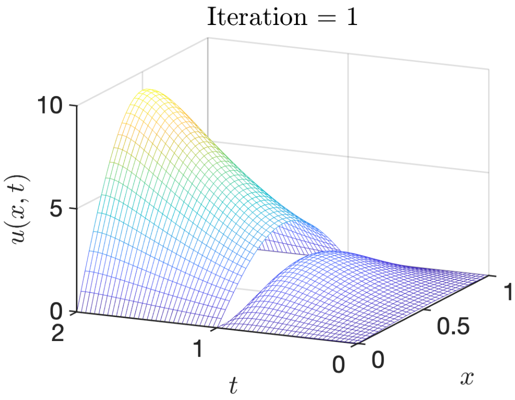

Weak scalability test

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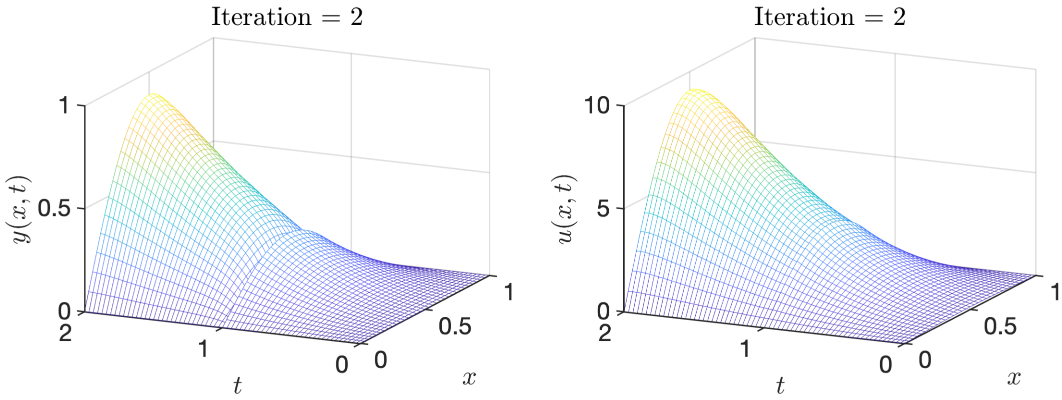

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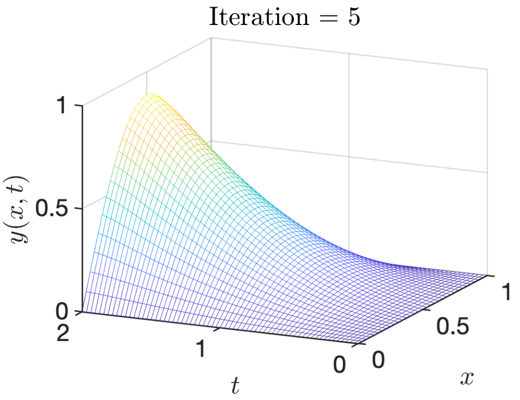
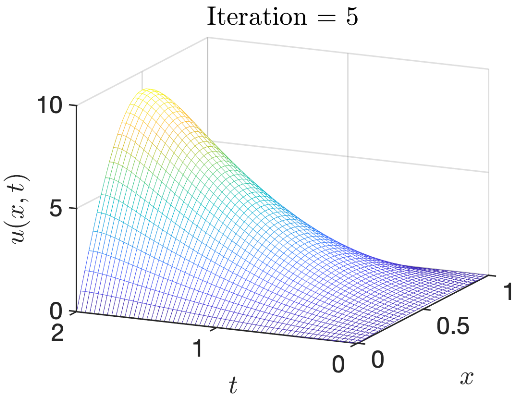
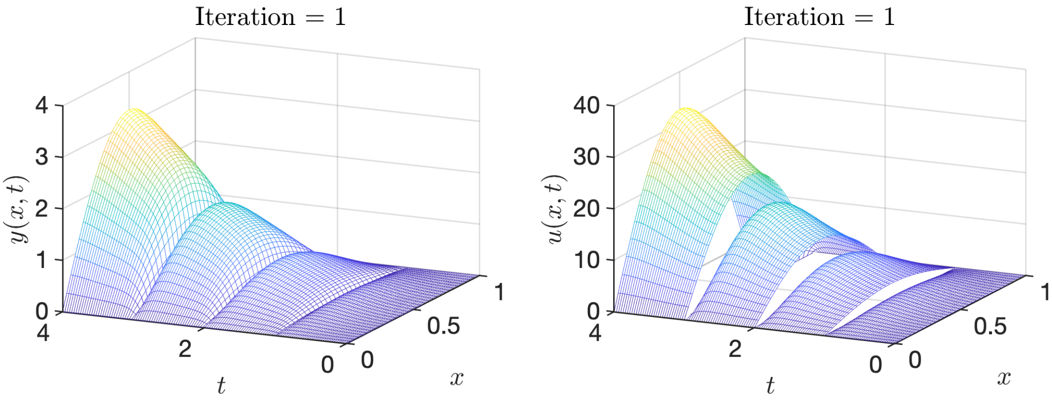




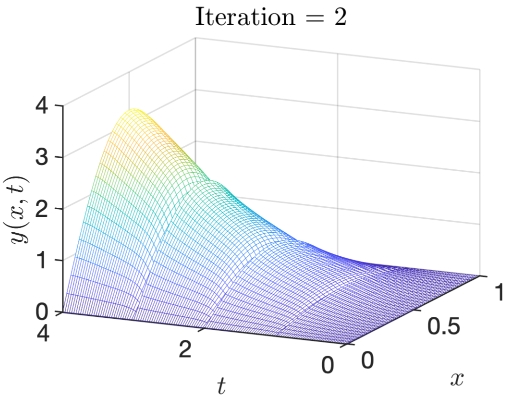
Weak scalability test

Four subdomains:

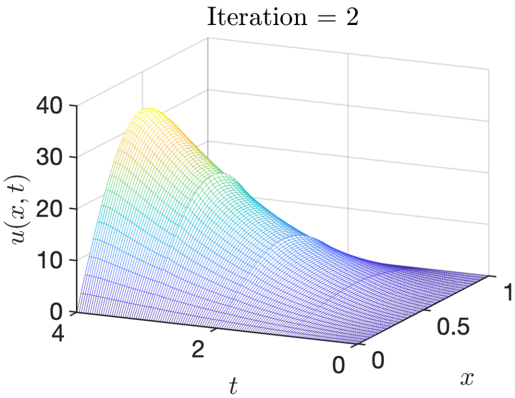


Weak scalability test

Four subdomains:



Iteration = 2



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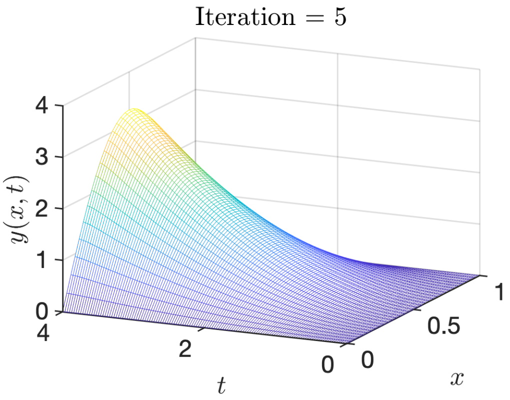
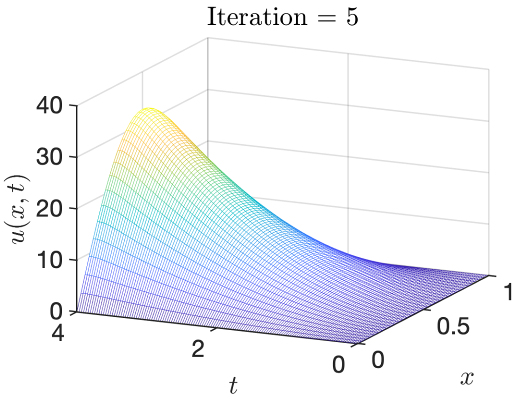
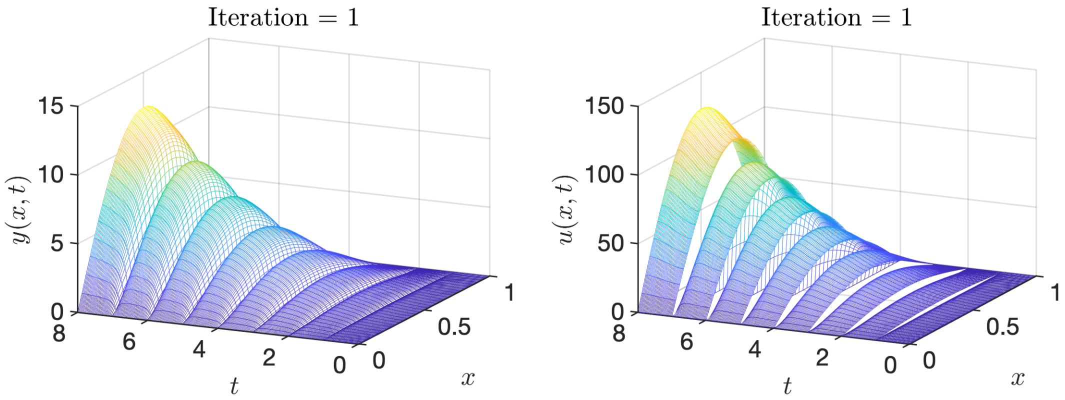




Weak scalability test

Eight subdomains:



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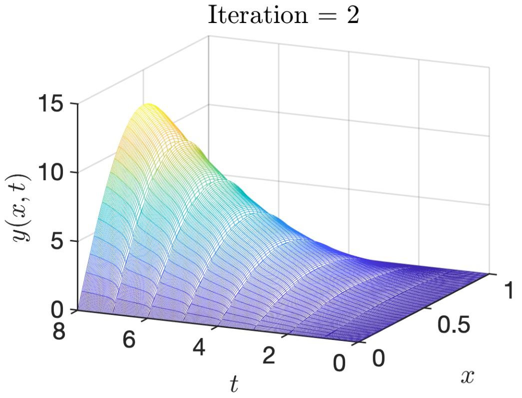

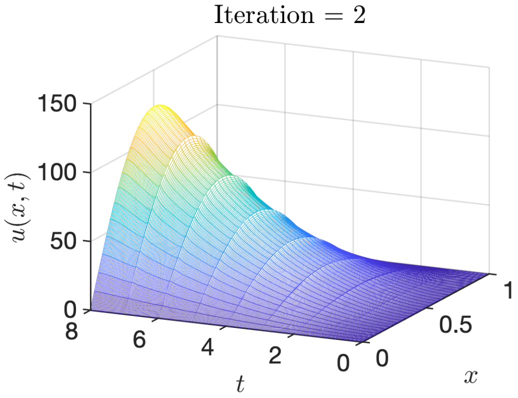

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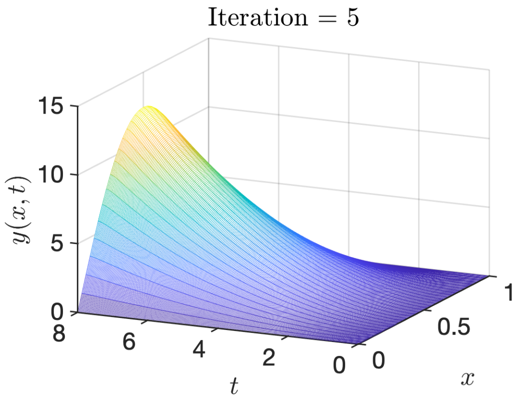
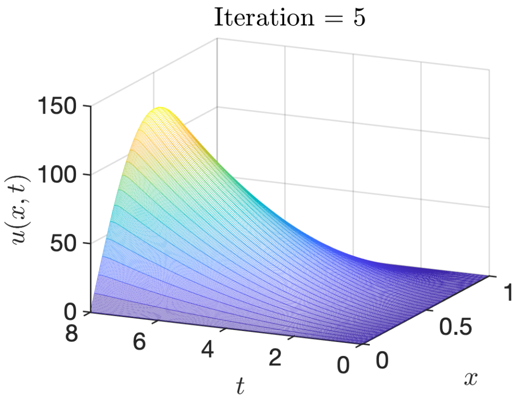
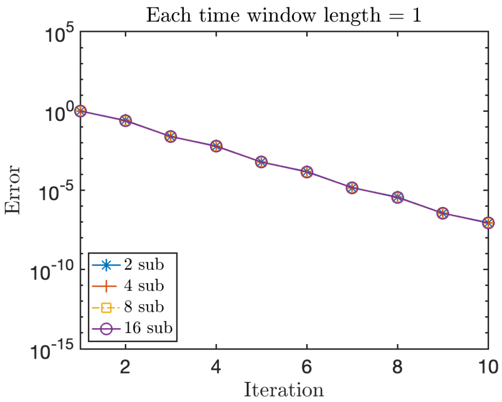
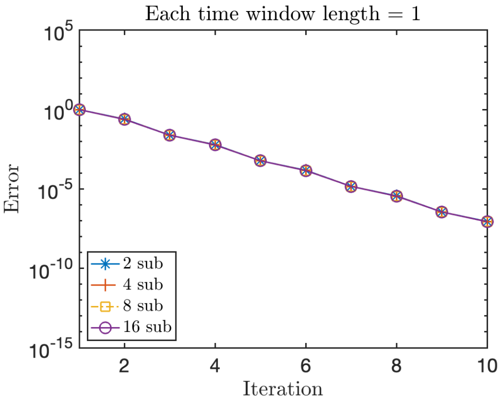




Error decay


Each time window length = 1
Error
Iteration
10^{5}
10^{0}
10^{-5}
10^{-10}
10^{-15}
2
4
6
8
10
2 sub
4 sub
8 sub
16 sub

Error decay


Each time window length = 1
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Iteration
10^5
10^0
10^-5
10^-10
10^-15
2
4
6
8
10
2 sub
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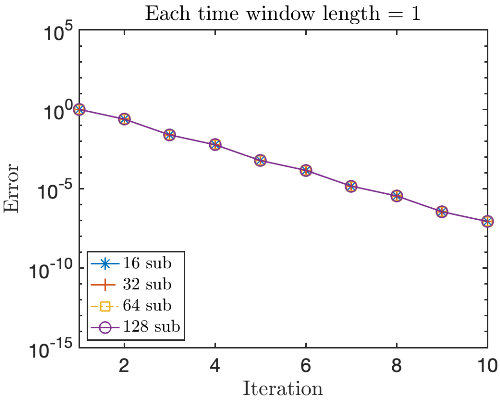
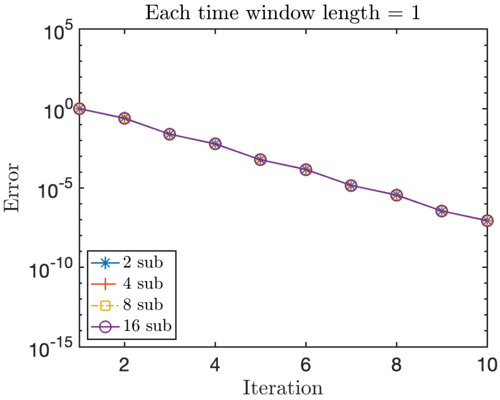
16 sub


Each time window length = 1
Error
Iteration
10^5
10^0
10^-5
10^-10
10^-15
2
4
6
8
10
16 sub
32 sub
64 sub
128 sub

Error decay


Each time window length = 1
Error
Iteration
10^5
10^0
10^-5
10^-10
10^-15
2
4
6
8
10
2 sub
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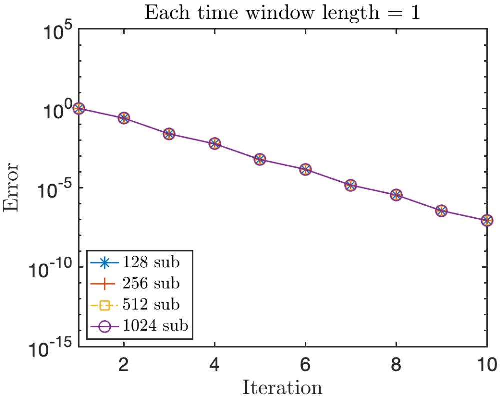
16 sub


Each time window length = 1
Error
Iteration
10^5
10^0
10^-5
10^-10
10^-15
2
4
6
8
10
128 sub
256 sub
512 sub
1024 sub

Extension to Navier–Stokes problems

Flow control w.r.t. some targets $\hat{\boldsymbol{y}} \in L^2(0, T; L^2(\Omega)^2)$

minimize the functional $J(\boldsymbol{y}, \boldsymbol{u}) := \frac{1}{2}\|\boldsymbol{y} - \hat{\boldsymbol{y}}\|^2_{L^2(0,T;L^2(\Omega)^2)} + \frac{\nu}{2}\|\boldsymbol{u}\|^2_{L^2(0,T;L^2(\Omega)^2)}$

constrained by

$$\partial_t \boldsymbol{y} - \kappa\Delta\boldsymbol{y} + (\boldsymbol{y}\cdot\nabla)\boldsymbol{y} + \nabla p = \boldsymbol{u} \text{ in } Q, \quad \operatorname{div}\boldsymbol{y} = 0 \text{ in } Q$$

completed with initial condition $\boldsymbol{y}_0$ and boundary condition $\boldsymbol{y} = 0$ on Σ

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completed with initial condition $\boldsymbol{y}_0$ and boundary condition $\boldsymbol{y} = 0$ on Σ

Lagrange multipliers $(\boldsymbol{\lambda}, q)$:

$$\mathcal{L}(\boldsymbol{y}, p, \boldsymbol{\lambda}, q, \boldsymbol{u}) = J(\boldsymbol{y}, \boldsymbol{u}) + \langle\boldsymbol{\lambda}, \partial_t\boldsymbol{y} + (\boldsymbol{y}\cdot\nabla)\boldsymbol{y} - \Delta\boldsymbol{y} + \nabla p - \boldsymbol{u}\rangle + \langle q, \operatorname{div}\boldsymbol{y}\rangle$$

Extension to Navier–Stokes problems

Flow control w.r.t. some targets $\hat{\boldsymbol{y}} \in L^2(0,T;L^2(\Omega)^2)$

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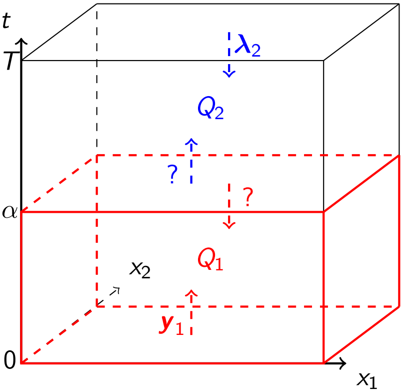
Derive first-order optimality system

$$\begin{aligned}
\partial_t\boldsymbol{y} - \kappa\Delta\boldsymbol{y} + (\boldsymbol{y}\cdot\nabla)\boldsymbol{y} + \nabla p &= \boldsymbol{u} && \text{in } Q & \partial_t\boldsymbol{\lambda} + \kappa\Delta\boldsymbol{\lambda} - (\nabla\boldsymbol{y})^T\boldsymbol{\lambda} + (\boldsymbol{y}\cdot\nabla)\boldsymbol{\lambda} + \nabla q &= \boldsymbol{y} - \hat{\boldsymbol{y}} && \text{in } Q \\
\operatorname{div}\boldsymbol{y} &= 0 && \text{in } Q & \operatorname{div}\boldsymbol{\lambda} &= 0 && \text{in } Q \\
\boldsymbol{y} &= \boldsymbol{y}_0 && \text{on } \Sigma_0 & \boldsymbol{\lambda} &= 0 && \text{on } \Sigma_T
\end{aligned}$$

$$\nu\boldsymbol{u} - \boldsymbol{\lambda} = 0 \quad \text{in } Q$$

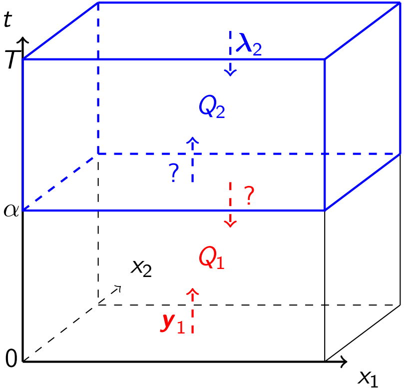
Apply time decomposition



In $Q_1 := \Omega \times (0, \alpha)$

$$
\begin{aligned}
\partial_t \boldsymbol{y}_1^\ell - \kappa \Delta \boldsymbol{y}_1^\ell + (\boldsymbol{y}_1^\ell \cdot \nabla) \boldsymbol{y}_1^\ell + \nabla p_1^\ell &= \boldsymbol{u}_1^\ell \\
\operatorname{div} \boldsymbol{y}_1^\ell &= 0 \\
\boldsymbol{y}_1^\ell|_0 &= \boldsymbol{y}_0 \\
\partial_t \boldsymbol{\lambda}_1^\ell + \kappa \Delta \boldsymbol{\lambda}_1^\ell - (\nabla \boldsymbol{y}_1^\ell)^T \boldsymbol{\lambda}_1^\ell + (\boldsymbol{y}_1^\ell \cdot \nabla) \boldsymbol{\lambda}_1^\ell + \nabla q_1^\ell &= \boldsymbol{y}_1^\ell - \hat{\boldsymbol{y}} \\
\operatorname{div} \boldsymbol{\lambda}_1^\ell &= 0 \\
\mathcal{T}_1(\boldsymbol{y}_1^\ell, \boldsymbol{\lambda}_1^\ell)|_\alpha &= \mathcal{T}_1(\boldsymbol{y}_2^{\ell-1}, \boldsymbol{\lambda}_2^{\ell-1})|_\alpha \\
\nu \boldsymbol{u}_1^\ell - \boldsymbol{\lambda}_1^\ell &= 0
\end{aligned}
$$

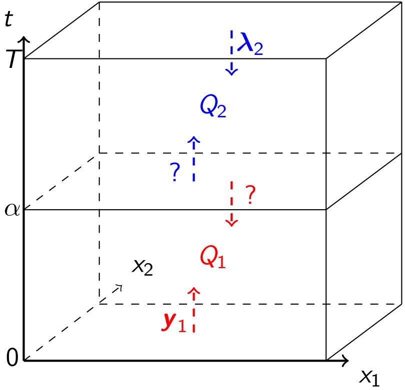
Apply time decomposition



In $Q_2 := \Omega \times (\alpha, T)$

$$\begin{aligned}
\partial_t \boldsymbol{y}_2^\ell - \kappa \Delta \boldsymbol{y}_2^\ell + (\boldsymbol{y}_2^\ell \cdot \nabla)\boldsymbol{y}_2^\ell + \nabla p_2^\ell &= \boldsymbol{u}_2^\ell \\
\operatorname{div} \boldsymbol{y}_2^\ell &= 0 \\
\mathcal{T}_2(\boldsymbol{y}_2^\ell, \boldsymbol{\lambda}_2^\ell)|_\alpha &= \mathcal{T}_2(\boldsymbol{y}_1^{\ell-1}, \boldsymbol{\lambda}_1^{\ell-1})|_\alpha \\
\partial_t \boldsymbol{\lambda}_2^\ell + \kappa \Delta \boldsymbol{\lambda}_2^\ell - (\nabla \boldsymbol{y}_2^\ell)^T \boldsymbol{\lambda}_2^\ell + (\boldsymbol{y}_2^\ell \cdot \nabla)\boldsymbol{\lambda}_2^\ell + \nabla q_2^\ell &= \boldsymbol{y}_2^\ell - \hat{\boldsymbol{y}} \\
\operatorname{div} \boldsymbol{\lambda}_2^\ell &= 0 \\
\boldsymbol{\lambda}_2^\ell|_T &= 0 \\
\nu \boldsymbol{u}_2^\ell - \boldsymbol{\lambda}_2^\ell &= 0
\end{aligned}$$

Simplify to Stokes problems



In $Q_1 = \Omega \times (0, \alpha)$

$$\partial_t \boldsymbol{y}_1^\ell - \kappa \Delta \boldsymbol{y}_1^\ell + \nabla p_1^\ell = \boldsymbol{u}_1^\ell \qquad \partial_t \boldsymbol{\lambda}_1^\ell + \kappa \Delta \boldsymbol{\lambda}_1^\ell + \nabla q_1^\ell = \boldsymbol{y}_1^\ell - \hat{\boldsymbol{y}}$$

$$\operatorname{div} \boldsymbol{y}_1^\ell = 0 \qquad \operatorname{div} \boldsymbol{\lambda}_1^\ell = 0$$

$$\boldsymbol{y}_1^\ell|_0 = \boldsymbol{y}_0 \quad \mathcal{T}_1(\boldsymbol{y}_1^\ell, \boldsymbol{\lambda}_1^\ell)|_\alpha = \mathcal{T}_1(\boldsymbol{y}_2^{\ell-1}, \boldsymbol{\lambda}_2^{\ell-1})|_\alpha$$

$$\nu \boldsymbol{u}_1^\ell - \boldsymbol{\lambda}_1^\ell = 0$$

In $Q_2 = \Omega \times (\alpha, T)$

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$$\nu \boldsymbol{u}_2^\ell - \boldsymbol{\lambda}_2^\ell = 0$$

Leray projection

For a vector field $\boldsymbol{v} : \mathbb{R}^d \to \mathbb{R}^d$

$$\mathbb{P}(\boldsymbol{v}) := \boldsymbol{v} - \nabla\Delta^{-1}(\operatorname{div}\boldsymbol{v})$$

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- Define projector $\mathbb{P} : L^2(\Omega)^d \to \mathcal{H}$ and Hilbert space $\mathcal{H}$

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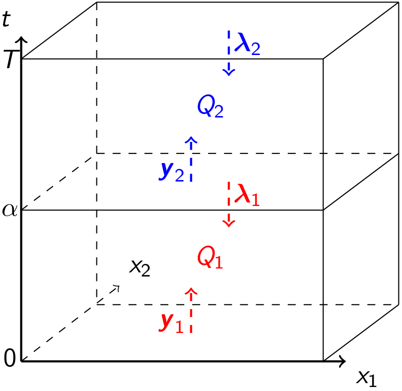
- Apply the spectral theorem to get $\{\mu_j, \psi_j(\boldsymbol{x})\}_j$

Convergence Stokes problems



Alternating Schwarz in time **does converge** !

In $Q_1 := \Omega \times (0, \alpha)$

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Conclusion

Some observations:

- Time decomposition is **different** from space decomposition
- Forward-backward structure is **important** for alternating Schwarz to converge without overlap
- Relaxation can improve the convergence, but is **not necessary** for the method to converge
- Alternating Schwarz in time is **weak** scalable (for heat problems)
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Thank you for your attention !