Non-overlapping domain decomposition methods for time parallel solution of PDE-constrained optimization problems

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UNIVERSITÉ DE GENÈVE

FACULTÉ DES SCIENCES
Section de mathématiques

Model problem

For $\hat{y} \in L^2(Q)$, $\gamma \geq 0$ and $\nu > 0$, minimize the cost functional

$$J(y,u) := \frac{1}{2}\|y - \hat{y}\|^2_{L^2(Q)} + \frac{\gamma}{2}\|y(T) - \hat{y}(T)\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)},$$

subject to

$$\begin{aligned} \partial_t y - \Delta y &= u && \text{in } Q := (0,T) \times \Omega, \\ y &= 0 && \text{on } \Sigma := (0,T) \times \partial\Omega, \\ y &= y_0 && \text{on } \Sigma_0 := \{0\} \times \Omega, \end{aligned}$$

with $\Omega \subset \mathbb{R}^n$.

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Lagrange multipliers method

$$\mathcal{L}(y,u,\lambda) = J(y,u) + \langle \partial_t y - \Delta y - u, \lambda \rangle.$$

First-order optimality system:

$$\begin{aligned} \partial_t y - \Delta y &= u \quad && \text{in } Q, \qquad & \partial_t \lambda + \Delta\lambda &= y - \hat{y} \quad && \text{in } Q, \\ y &= 0 && \text{in } \Sigma, & \lambda &= 0 && \text{in } \Sigma, \\ y &= y_0 && \text{in } \Sigma_0, & \lambda &= -\gamma(y - \hat{y}) && \text{in } \Sigma_T := \{T\}\times\Omega, \end{aligned}$$

$$-\lambda + \nu u = 0 \quad \text{in } Q.$$

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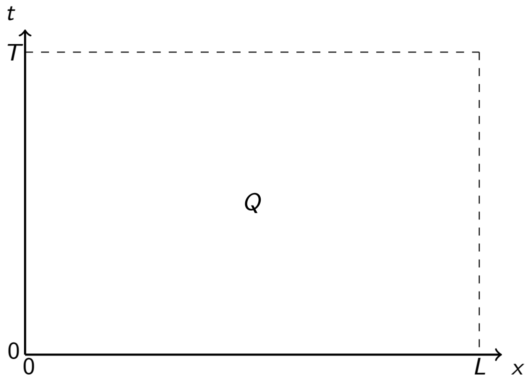
Lagrange multipliers method

$$\mathcal{L}(y,u,\lambda) = J(y,u) + \langle \partial_t y - \Delta y - u, \lambda \rangle.$$

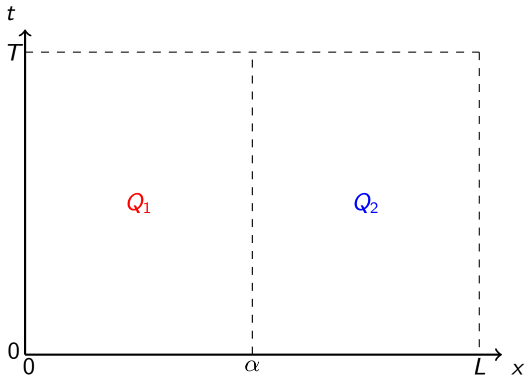
Reduced optimality system (forward-backward):

$$\begin{aligned} \partial_t y - \Delta y &= \nu^{-1}\lambda \quad && \text{in } Q, \qquad & \partial_t \lambda + \Delta \lambda &= y - \hat{y} \quad && \text{in } Q, \\ y &= 0 && \text{in } \Sigma, & \lambda &= 0 && \text{in } \Sigma, \\ y &= y_0 && \text{in } \Sigma_0, & \lambda &= -\gamma(y - \hat{y}) && \text{in } \Sigma_T. \end{aligned}$$



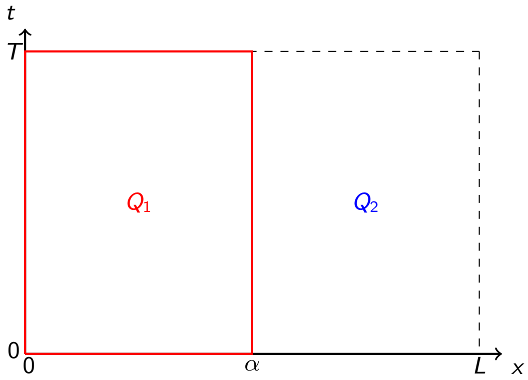
Space-time domain: $Q = (0, L) \times (0, T)$,

$$\begin{aligned} \partial_t y - \partial_{xx} y &= \nu^{-1}\lambda, & \partial_t \lambda + \partial_{xx}\lambda &= y - \hat{y}, \\ y(0,t) &= 0, & \lambda(0,t) &= 0, \\ y(L,t) &= 0, & \lambda(L,t) &= 0, \\ y(x,0) &= y_0(x), & \lambda(x,T) + \gamma y(x,T) &= \gamma \hat{y}(x,T), \end{aligned}$$



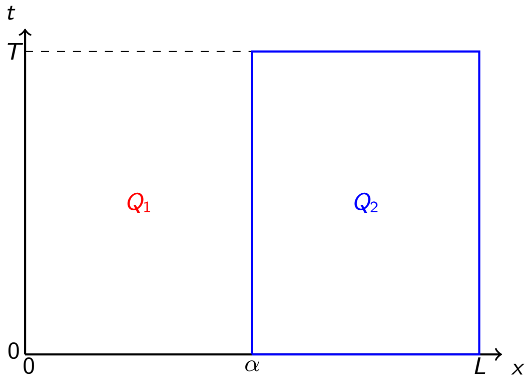
Subdomains: $Q_1 = (0, \alpha) \times (0, T)$ and $Q_2 = (\alpha, L) \times (0, T)$,

$$\begin{aligned} \partial_t y - \partial_{xx} y &= \nu^{-1}\lambda, & \partial_t \lambda + \partial_{xx}\lambda &= y - \hat{y}, \\ y(0,t) &= 0, & \lambda(0,t) &= 0, \\ y(L,t) &= 0, & \lambda(L,t) &= 0, \\ y(x,0) &= y_0(x), & \lambda(x,T) + \gamma y(x,T) &= \gamma \hat{y}(x,T), \end{aligned}$$



Subdomain: $Q_1 = (0, \alpha) \times (0, T)$,

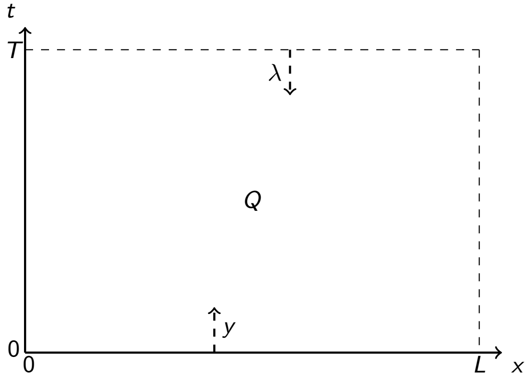
$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell, & \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell &= y_1^\ell - \hat{y}_1,\\
y_1^\ell(0,t) &= 0, & \lambda_1^\ell(0,t) &= 0,\\
y_1^\ell(\alpha,t) &= y_2^{\ell-1}(\alpha,t), & \lambda_1^\ell(\alpha,t) &= \lambda_2^{\ell-1}(\alpha,t),\\
y_1^\ell(x,0) &= y_{1,0}(x), & \lambda_1^\ell(x,T) + \gamma y_1^\ell(x,T) &= \gamma \hat{y}_1(x,T).
\end{aligned}
$$



Subdomains: $Q_2 = (\alpha, 1) \times (0, T)$,

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1}\lambda_2^\ell, & \partial_t \lambda_2^\ell + \partial_{xx}\lambda_2^\ell &= y_2^\ell - \hat{y}_2, \\ \partial_x y_2^\ell(\alpha, t) &= \partial_x y_1^\ell(\alpha, t), & \partial_x \lambda_2^\ell(\alpha, t) &= \partial_x \lambda_1^\ell(\alpha, t), \\ y_2^\ell(L, t) &= 0, & \lambda_2^\ell(L, t) &= 0, \\ y_2^\ell(x, 0) &= y_{2,0}(x), & \lambda_2^\ell(x, T) + \gamma y_2^\ell(x, T) &= \gamma \hat{y}_2(x, T). \end{aligned}$$

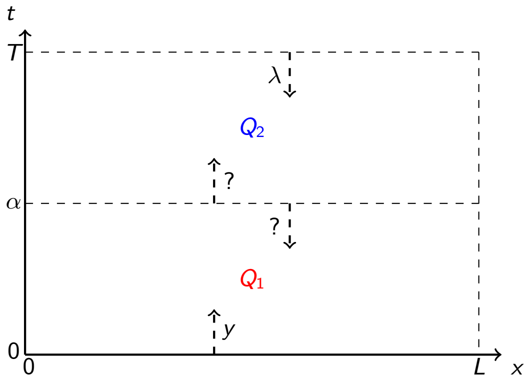
Time domain decomposition



Space-time domain: $Q = (0, L) \times (0, T)$,

$$
\begin{aligned}
\partial_t y - \partial_{xx} y &= \nu^{-1}\lambda, & \partial_t \lambda + \partial_{xx}\lambda &= y - \hat{y},\\
y(0,t) &= 0, & \lambda(0,t) &= 0,\\
y(L,t) &= 0, & \lambda(L,t) &= 0,\\
y(x,0) &= y_0(x), & \lambda(x,T) + \gamma y(x,T) &= \gamma \hat{y}(x,T).
\end{aligned}
$$

Time domain decomposition



Subdomains: $Q_1 = (0, L) \times (0, \alpha)$ and $Q_2 = (0, L) \times (\alpha, T)$,

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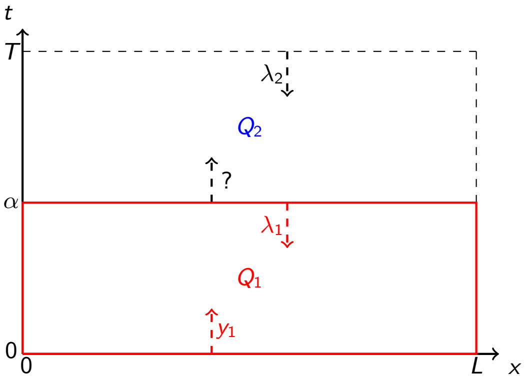
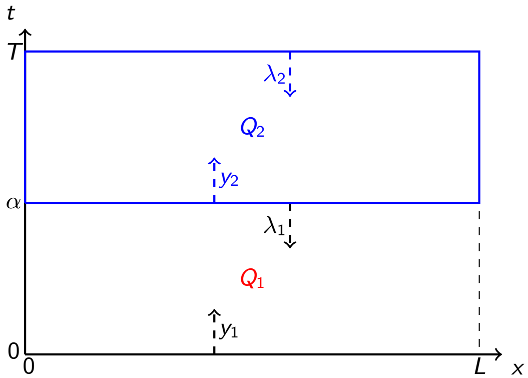
Time domain decomposition



Subdomain: $Q_1 = (0, L) \times (0, \alpha)$,

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell, & \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell &= y_1^\ell - \hat{y}_1, \\ y_1^\ell(0,t) &= 0, & \lambda_1^\ell(0,t) &= 0, \\ y_1^\ell(L,t) &= 0, & \lambda_1^\ell(L,t) &= 0, \\ y_1^\ell(x,0) &= y_0(x), & \lambda_1^\ell(x,\alpha) &= \lambda_2^{\ell-1}(x,\alpha), \end{aligned}$$



Subdomain: $Q_2 = (0, L) \times (\alpha, T)$,

$$
\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} \lambda_2^\ell, & \partial_t \lambda_2^\ell + \partial_{xx} \lambda_2^\ell &= y_2^\ell - \hat{y}_2, \\
y_2^\ell(0, t) &= 0, & \lambda_2^\ell(0, t) &= 0, \\
y_2^\ell(L, t) &= 0, & \lambda_2^\ell(L, t) &= 0, \\
\partial_t y_2^\ell(x, \alpha) &= \partial_t y_1^\ell(x, \alpha), & \lambda_2^\ell(x, T) + \gamma y_2^\ell(x, T) &= \gamma \hat{y}_2(x, T).
\end{aligned}
$$

Semi-discretization

Reduced optimality system:

$$\begin{cases} \partial_t \begin{pmatrix} y \\ \lambda \end{pmatrix} + \begin{pmatrix} -\partial_{xx} y - \nu^{-1}\lambda \\ -y + \partial_{xx}\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{y} \end{pmatrix}, \\ y(\cdot, 0) = y_0(\cdot), \\ \lambda(\cdot, T) + \gamma y(\cdot, T) = \gamma \hat{y}(\cdot, T), \end{cases} \quad \text{in } (0, L) \times (0, T)$$

Semi-discretization

Finite difference discretization: $-\partial_{xx} \approx A \in \mathbb{R}^{n\times n}$,

$$\begin{cases} \begin{pmatrix} \dot{\boldsymbol{y}} \\ \dot{\boldsymbol{\lambda}} \end{pmatrix} + \begin{pmatrix} A & -\nu^{-1} I \\ -I & -A \end{pmatrix} \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{\boldsymbol{y}} \end{pmatrix} \text{ in } (0, T), \\ \boldsymbol{y}(0) = \boldsymbol{y}_0, \\ \boldsymbol{\lambda}(T) + \gamma \boldsymbol{y}(T) = \gamma \hat{\boldsymbol{y}}(T), \end{cases}$$

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Diagonalization: $A = PDP^{-1}$ and $D = \mathrm{diag}(d_1, \ldots, d_n)$,

$$\begin{cases} \begin{pmatrix} \dot{z}_i \\ \dot{\mu}_i \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_i \\ \mu_i \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{z}_i \end{pmatrix} \text{ in } (0, T), \\ z_i(0) = z_{0,i}, \\ \mu_i(T) + \gamma z_i(T) = \gamma \hat{z}_i(T), \end{cases}$$

with $\boldsymbol{z} = P^{-1}\boldsymbol{y}$, $\hat{\boldsymbol{z}} = P^{-1}\hat{\boldsymbol{y}}$ and $\boldsymbol{\mu} = P^{-1}\boldsymbol{\lambda}$. So n **independent** 2×2 **systems**.

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Second-order ODE:

$$\begin{cases} \ddot{z}_i - \sigma_i^2 z_i = -\nu^{-1}\hat{z}_i \text{ in } (0, T), \\ z_i(0) = z_{0,i}, \\ \dot{z}_i(T) + \omega_i z_i(T) = \nu^{-1}\gamma \hat{z}_i(T), \end{cases}$$

with $\sigma_i := \sqrt{d_i^2 + \nu^{-1}}$, $\omega_i := \nu^{-1}\gamma + d_i$ and $\beta_i := 1 - \gamma d_i$.

Finite difference discretization: $-\partial_{xx} \approx A \in \mathbb{R}^{n\times n}$,

$$\begin{cases} \begin{pmatrix} \dot{\boldsymbol{y}} \\ \dot{\boldsymbol{\lambda}} \end{pmatrix} + \begin{pmatrix} A & -\nu^{-1} I \\ -I & -A \end{pmatrix} \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{\boldsymbol{y}} \end{pmatrix} \text{ in } (0, T), \\ \boldsymbol{y}(0) = \boldsymbol{y}_0, \\ \boldsymbol{\lambda}(T) + \gamma \boldsymbol{y}(T) = \gamma \hat{\boldsymbol{y}}(T), \end{cases}$$

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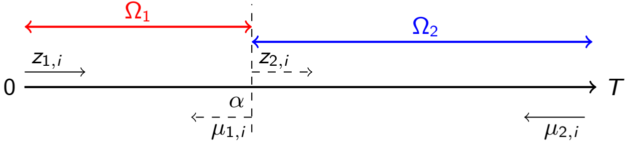
with $\boldsymbol{z} = P^{-1}\boldsymbol{y}$, $\hat{\boldsymbol{z}} = P^{-1}\hat{\boldsymbol{y}}$ and $\boldsymbol{\mu} = P^{-1}\boldsymbol{\lambda}$. So n **independent** 2×2 **systems**.
Second-order ODE:

$$\begin{cases} \ddot{\mu}_i - \sigma_i^2 \mu_i = -(\dot{\hat{z}}_i + d_i \hat{z}_i) \text{ in } (0, T), \\ \mu_i(0) - d_i \mu_i(0) = z_{0,i} - \hat{z}_i(0), \\ \gamma \dot{\mu}_i(T) + \beta_i \mu_i(T) = 0, \end{cases}$$

with $\sigma_i := \sqrt{d_i^2 + \nu^{-1}}$, $\omega_i := \nu^{-1}\gamma + d_i$ and $\beta_i := 1 - \gamma d_i$.



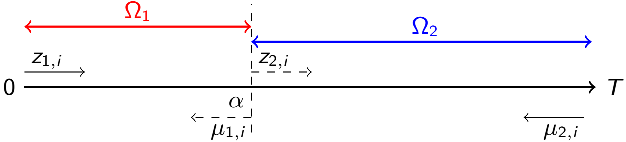
Dirichlet:

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^{\ell} \\ \dot{\mu}_{1,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^{\ell} \\ \mu_{1,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ \mu_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases}$$

Update:

$$f_{\alpha,i}^{\ell} := (1-\theta) f_{\alpha,i}^{\ell-1} + \theta \mu_{2,i}^{\ell}(\alpha), \quad \theta \in (0,1).$$

Transformation



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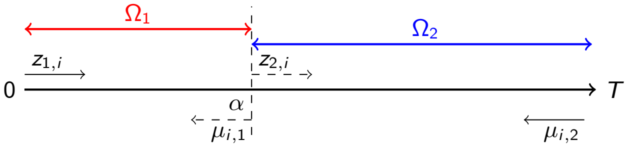
Neumann:

$$\begin{cases} \begin{pmatrix} \dot{z}_{2,i}^{\ell} \\ \dot{\mu}_{2,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^{\ell} \\ \mu_{2,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}_{2,i}^{\ell}(\alpha) = \dot{z}_{1,i}^{\ell}(\alpha), \\ \mu_{2,i}^{\ell}(T) + \gamma z_{2,i}^{\ell}(T) = 0, \end{cases}$$

Update:

$$f_{\alpha,i}^{\ell} := (1-\theta) f_{\alpha,i}^{\ell-1} + \theta \mu_{2,i}^{\ell}(\alpha), \quad \theta \in (0,1).$$

Transformation



Dirichlet:

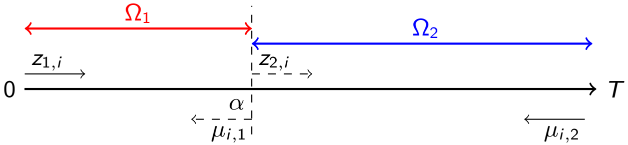
$$\begin{cases} \ddot{z}_{1,i}^{\ell} - \sigma_i^2 z_{1,i}^{\ell} = 0 \text{ in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ \dot{z}_{1,i}^{\ell}(\alpha) + d_i z_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases}$$

Neumann:

$$\begin{cases} \ddot{z}_{2,i}^{\ell} - \sigma_i^2 z_{2,i}^{\ell} = 0 \text{ in } \Omega_2, \\ \dot{z}_{2,i}^{\ell}(\alpha) = \dot{z}_{1,i}^{\ell}(\alpha), \\ \dot{z}_{2,i}^{\ell}(T) + \omega_i z_{2,i}^{\ell}(T) = 0, \end{cases}$$

Update:

$$f_{\alpha,i}^{\ell} = (1-\theta) f_{\alpha,i}^{\ell-1} + \theta\big(\dot{z}_{2,i}^{\ell}(\alpha) + d_i z_{2,i}^{\ell}(\alpha)\big), \quad \theta \in (0,1)$$



Dirichlet:

$$\begin{cases} \ddot{\mu}_{1,i}^{\ell} - \sigma_i^2 \mu_{1,i}^{\ell} = 0 \text{ in } \Omega_1, \\ \dot{\mu}_i(0) - d_i \mu_i(0) = 0, \\ \mu_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases}$$

Neumann:

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Update:

$$f_{\alpha,i}^{\ell} = (1-\theta) f_{\alpha,i}^{\ell-1} + \theta \mu_{2,i}^{\ell}(\alpha), \quad \theta \in (0,1).$$

Transformation

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^{\ell} \\ \dot{\mu}_{1,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^{\ell} \\ \mu_{1,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ \mu_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases}$$

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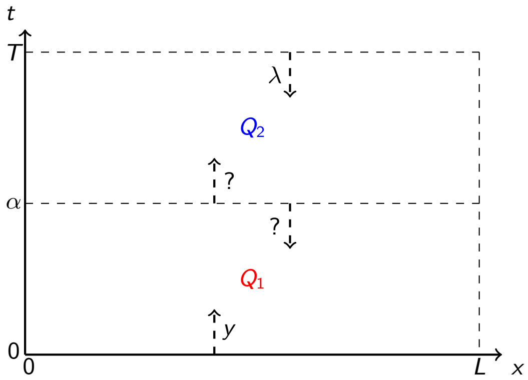
$$f_{\alpha,i}^{\ell} := (1-\theta) f_{\alpha,i}^{\ell-1} + \theta \mu_{2,i}^{\ell}(\alpha), \quad \theta \in (0,1).$$

Two observations:

(1) Three systems are equivalent, so same convergence using z or μ;

(2) Not anymore a "DN" algorithm, and forward-backward structure is less important.



Space-time domain: $Q = (0, L) \times (0, T)$,

$$\begin{aligned} \partial_t y - \Delta y &= \nu^{-1}\lambda, & \partial_t \lambda + \Delta \lambda &= y - \hat{y}, \\ y(0,t) &= 0, & \lambda(0,t) &= 0, \\ y(L,t) &= 0, & \lambda(L,t) &= 0, \\ y(x,0) &= y_0(x), & \lambda(x,T) + \gamma y(x,T) &= \gamma \hat{y}(x,T). \end{aligned}$$

Variants

Dirichlet–Neumann variants ($y \sim z_i$ and $\lambda \sim \mu_i$)

name	Ω_1	Ω_2	type
DN_1	μ_i	$\dot{z}_i$	(DN)
	$\dot{z}_i + d_i z_i$	$\dot{z}_i$	(RN)
DN_2	z_i	$\dot{z}_i$	(DN)
	z_i	$\dot{z}_i$	(DN)
DN_3	μ_i	$\dot{\mu}_i$	(DN)
	$\dot{z}_i + d_i z_i$	$\ddot{z}_i + d_i \dot{z}_i$	(RR)

Variants

Neumann–Dirichlet variants ($y \sim z_i$ and $\lambda \sim \mu_i$)

name	Ω_1	Ω_2	type
ND_1	$\dot{\mu}_i$	z_i	(ND)
	$\ddot{z}_i + d_i \dot{z}_i$	z_i	(RD)
ND_2	$\dot{z}_i$	z_i	(ND)
	$\dot{z}_i$	z_i	(ND)
ND_3	$\dot{\mu}_i$	μ_i	(ND)
	$\ddot{z}_i + d_i \dot{z}_i$	$\dot{z}_i + d_i z_i$	(RR)

Comparaison of two DN variants

Natural Dirichlet–Neumann (DN_1):

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^{\ell} \\ \dot{\mu}_{1,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^{\ell} \\ \mu_{1,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ \mu_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases} \quad \begin{cases} \begin{pmatrix} \dot{z}_{2,i}^{\ell} \\ \dot{\mu}_{2,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^{\ell} \\ \mu_{2,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}_{2,i}^{\ell}(\alpha) = \dot{z}_{1,i}^{\ell}(\alpha), \\ \mu_{2,i}^{\ell}(T) + \gamma z_{2,i}^{\ell}(T) = 0, \end{cases}$$

$$f_{\alpha,i}^{\ell} := (1-\theta) f_{\alpha,i}^{\ell-1} + \theta \mu_{2,i}^{\ell}(\alpha).$$

Dirichlet–Neumann at two levels (DN_2):

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^{\ell} \\ \dot{\mu}_{1,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^{\ell} \\ \mu_{1,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ z_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases} \quad \begin{cases} \begin{pmatrix} \dot{z}_{2,i}^{\ell} \\ \dot{\mu}_{2,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^{\ell} \\ \mu_{2,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}_{2,i}^{\ell}(\alpha) = \dot{z}_{1,i}^{\ell}(\alpha), \\ \mu_{2,i}^{\ell}(T) + \gamma z_{2,i}^{\ell}(T) = 0, \end{cases}$$

$$f_{\alpha,i}^{\ell} := (1-\theta) f_{\alpha,i}^{\ell-1} + \theta z_{2,i}^{\ell}(\alpha).$$

Forward–backward structure can always be recovered !

$$z_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1} \Rightarrow \dot{\mu}_{1,i}^{\ell}(\alpha) - d_i \mu_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}.$$

Comparaison of two DN variants

Natural Dirichlet–Neumann (DN_1):

$$\begin{cases} \ddot{z}_{1,i}^{\ell} - \sigma_i^2 z_{1,i}^{\ell} = 0 \quad \text{in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ \dot{z}_{1,i}^{\ell}(\alpha) + d_i z_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases} \qquad \begin{cases} \ddot{z}_{2,i}^{\ell} - \sigma_i^2 z_{2,i}^{\ell} = 0 \quad \text{in } \Omega_2, \\ \dot{z}_{2,i}^{\ell}(\alpha) = \dot{z}_{1,i}^{\ell}(\alpha), \\ \dot{z}_{2,i}^{\ell}(T) + \omega_i z_{2,i}^{\ell}(T) = 0, \end{cases}$$

$$f_{\alpha,i}^{\ell} := (1-\theta) f_{\alpha,i}^{\ell-1} + \theta\big(\dot{z}_{2,i}^{\ell}(\alpha) + d_i z_{2,i}^{\ell}(\alpha)\big), \quad \theta \in (0,1).$$

Dirichlet–Neumann at two levels (DN_2):

$$\begin{cases} \ddot{z}_{1,i}^{\ell} - \sigma_i^2 z_{1,i}^{\ell} = 0 \quad \text{in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ z_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases} \qquad \begin{cases} \ddot{z}_{2,i}^{\ell} - \sigma_i^2 z_{2,i}^{\ell} = 0 \quad \text{in } \Omega_2, \\ \dot{z}_{2,i}^{\ell}(\alpha) = \dot{z}_{1,i}^{\ell}(\alpha), \\ \dot{z}_{2,i}^{\ell}(T) + \omega_i z_{2,i}^{\ell}(T) = 0, \end{cases}$$

$$f_{\alpha,i}^{\ell} := (1-\theta) f_{\alpha,i}^{\ell-1} + \theta z_{2,i}^{\ell}(\alpha), \quad \theta \in (0,1).$$

Comparaison of two DN variants

Solve the problem and find

$$f_{\alpha,i}^{\ell} = \rho(\alpha, d_i, \nu, \gamma, \theta)\, f_{\alpha,i}^{\ell-1}.$$

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Convergence factor with analytical form

$$\rho_{\mathrm{DN}_1} := \max_{d_i \in \lambda(A)} \left| 1 - \theta\left(1 - \nu^{-1} \frac{\sigma_i \gamma + \beta_i \tanh(b_i)}{\left(\sigma_i + d_i \tanh(a_i)\right)\left(\omega_i + \sigma_i \tanh(b_i)\right)}\right) \right|,$$

$$\rho_{\mathrm{ND}_1} := \max_{d_i \in \lambda(A)} \left| 1 - \theta\left(1 - \nu^{-1} \frac{\sigma_i \gamma + \beta_i \coth(b_i)}{\left(\sigma_i + d_i \coth(a_i)\right)\left(\omega_i + \sigma_i \coth(b_i)\right)}\right) \right|.$$

Comparaison of two DN variants

Solve the problem and find

$$f_{\alpha,i}^{\ell} = \rho(\alpha, d_i, \nu, \gamma, \theta)\, f_{\alpha,i}^{\ell-1}.$$

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$$\rho_{\mathrm{DN}_1} := \max_{d_i \in \lambda(A)} \left| 1 - \theta\left(1 - \nu^{-1} \frac{\sigma_i \gamma + \beta_i \tanh(b_i)}{(\sigma_i + d_i \tanh(a_i))\,(\omega_i + \sigma_i \tanh(b_i))}\right)\right|,$$

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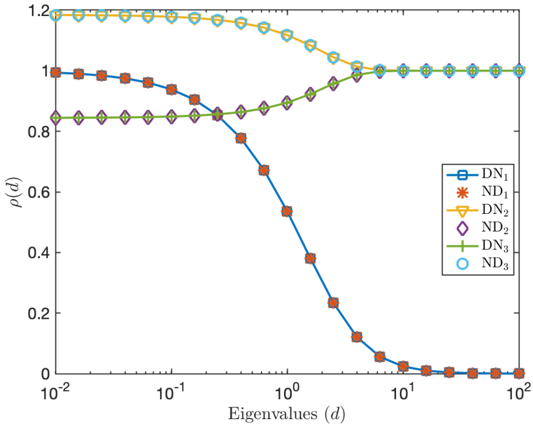
"Optimal" relaxation parameter with equioscillation

$$\theta^*_{\mathrm{DN}_2} = \frac{2}{3 + \coth(\sqrt{\nu^{-1}}\alpha)\frac{\coth\left(\sqrt{\nu^{-1}}(T-\alpha)\right) + \gamma\sqrt{\nu^{-1}}}{1 + \gamma\sqrt{\nu^{-1}}\coth\left(\sqrt{\nu^{-1}}(T-\alpha)\right)}},$$

$$\theta^*_{\mathrm{ND}_2} = \frac{2}{3 + \tanh(\sqrt{\nu^{-1}}\alpha)\frac{\tanh\left(\sqrt{\nu^{-1}}(T-\alpha)\right) + \gamma\sqrt{\nu^{-1}}}{1 + \gamma\sqrt{\nu^{-1}}\tanh\left(\sqrt{\nu^{-1}}(T-\alpha)\right)}},$$

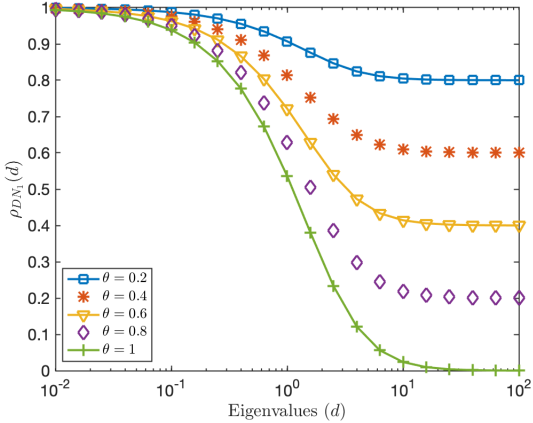
$\theta^*_{\mathrm{DN}_2} = \theta^*_{\mathrm{ND}_3}$ and $\theta^*_{\mathrm{ND}_2} = \theta^*_{\mathrm{DN}_3}$.

Convergence factor of different DN and ND variants, with penalization parameters: $\nu = 0.1$, $\gamma = 0$, interface: $\alpha = \frac{T}{2}$, and relaxation parameter: $\theta = 1$.



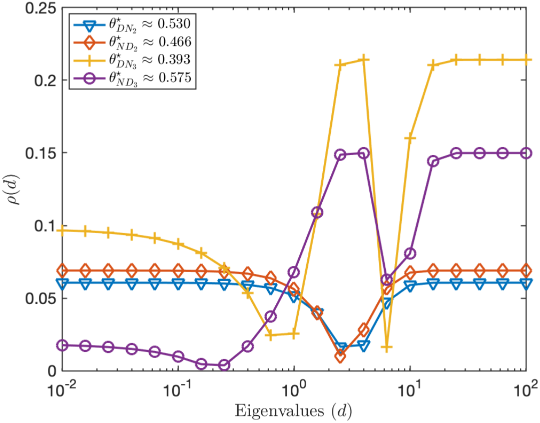
Numerical experiments

Convergence factor of different DN_1, with penalization parameters: $\nu = 0.1$, $\gamma = 0$, interface: $\alpha = \frac{T}{2}$.



Numerical experiments

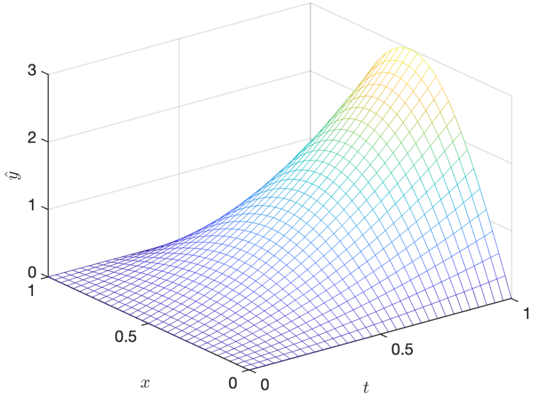
Optimal convergence factors with penalization parameters: $\nu = 0.1$, $\gamma = 10$ and interface: $\alpha = \frac{7}{10}T$.



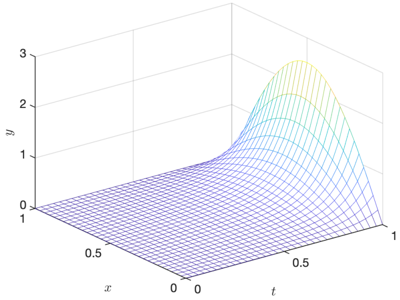
$\theta^{\star}_{\mathrm{DN}_2} = \theta^{*}_{\mathrm{DN}_2}$ and $\theta^{\star}_{\mathrm{ND}_2} = \theta^{*}_{\mathrm{ND}_2}$.

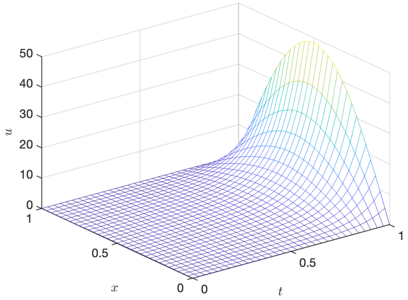
Numerical experiments

Numerical tests with penalization parameters: $\nu = 0.1$, $\gamma = 10$, final time: $T = 1$, and a target function $\hat{y}(t, x) = \sin(\pi x)(2t^2 + t)$.

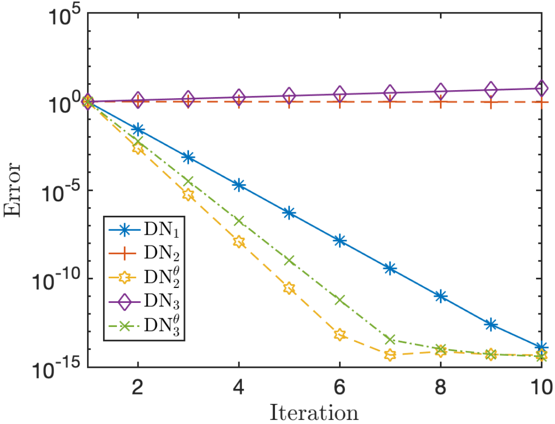


Numerical scheme: Crank-Nicolson, and mesh size: $h_t = h_x = \frac{1}{32}$.





With an interface $\alpha = \frac{7}{10} T$.



Conclusion

- Time domain decomposition is different from space domain decomposition.
- Forward-backward structure is less important for Dirichlet–Neumann method.

Gander and Lu, *New time domain decomposition methods for parabolic optimal control problems I: Dirichlet–Neumann and Neumann–Dirichlet algorithms*, SIAM J. Numer. Anal., 62(4):2048–2070 (2024)

Gander and Lu, *New time domain decomposition methods for parabolic optimal control problems II: Neumann–Neumann algorithms*, SIAM J. Numer. Anal., 62(6):2588–2610 (2024)

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- This analysis can be used for higher dimension and for other constraint equations, e.g., Stokes.
- Like always, there are a lot of interesting things to be further discovered ;)

Thanks for your attention !