Time domain decomposition and application to PDE-constrained optimization problems

Liu-Di LU

Section of Mathematics
University of Geneva

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Joint work with Martin J. Gander

Model problem

For $\hat{y} \in L^2(Q)$, $\gamma \geq 0$ and $\nu > 0$, minimize the cost functional

$$J(y,u) := \frac{1}{2}\|y - \hat{y}\|^2_{L^2(Q)} + \frac{\gamma}{2}\|y(T) - \hat{y}(T)\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)},$$

subject to

$$\begin{aligned} \partial_t y - \Delta y &= u && \text{in } Q := (0,T) \times \Omega, \\ y &= 0 && \text{on } \Sigma := (0,T) \times \partial\Omega, \\ y &= y_0 && \text{on } \Sigma_0 := \{0\} \times \Omega, \end{aligned}$$

with $\Omega \subset \mathbb{R}^n$.

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Lagrange multipliers method

$$\mathcal{L}(y,u,\lambda) = J(y,u) + \langle \partial_t y - \Delta y - u, \lambda \rangle.$$

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First-order optimality system:

$$\begin{aligned} \partial_t y - \Delta y &= u && \text{in } Q, & \partial_t \lambda + \Delta\lambda &= y - \hat{y} && \text{in } Q, \\ y &= 0 && \text{in } \Sigma, & \lambda &= 0 && \text{in } \Sigma, \\ y &= y_0 && \text{in } \Sigma_0, & \lambda &= -\gamma(y - \hat{y}) && \text{in } \Sigma_T := \{T\}\times\Omega, \end{aligned}$$

$$-\lambda + \nu u = 0 \quad \text{in } Q.$$

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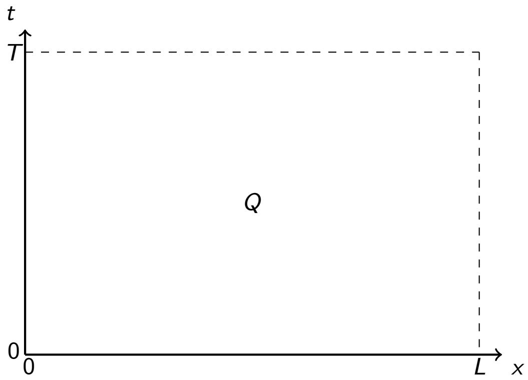
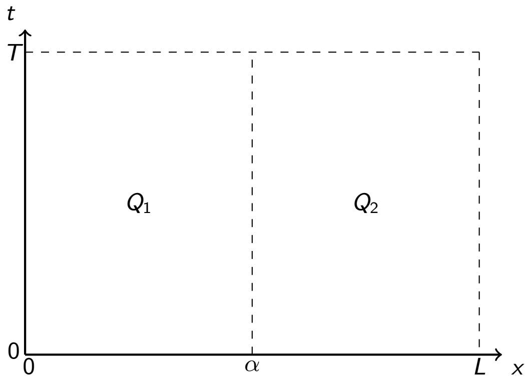
Reduced optimality system (forward-backward):

$$\begin{aligned} \partial_t y - \Delta y &= \nu^{-1}\lambda && \text{in } Q, & \partial_t \lambda + \Delta \lambda &= y - \hat{y} && \text{in } Q, \\ y &= 0 && \text{in } \Sigma, & \lambda &= 0 && \text{in } \Sigma, \\ y &= y_0 && \text{in } \Sigma_0, & \lambda &= -\gamma(y - \hat{y}) && \text{in } \Sigma_T. \end{aligned}$$



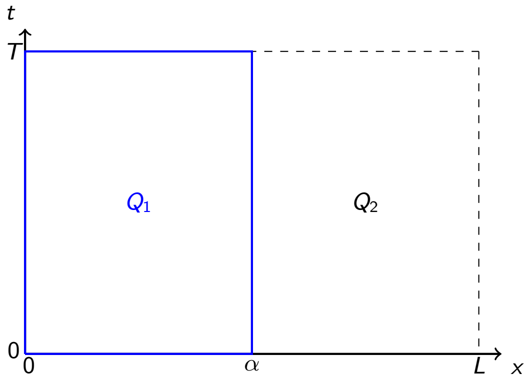
$Q = (0, L) \times (0, T)$,

$$\begin{aligned} \partial_t y - \Delta y &= \nu^{-1}\lambda, & \partial_t \lambda + \Delta \lambda &= y - \hat{y}, \\ y(0,t) &= 0, & \lambda(0,t) &= 0, \\ y(L,t) &= 0, & \lambda(L,t) &= 0, \\ y(x,0) &= y_0(x), & \lambda(x,T) + \gamma y(x,T) &= \gamma \hat{y}(x,T). \end{aligned}$$



$Q_1 = (0, \alpha) \times (0, T)$ and $Q_2 = (\alpha, L) \times (0, T)$,

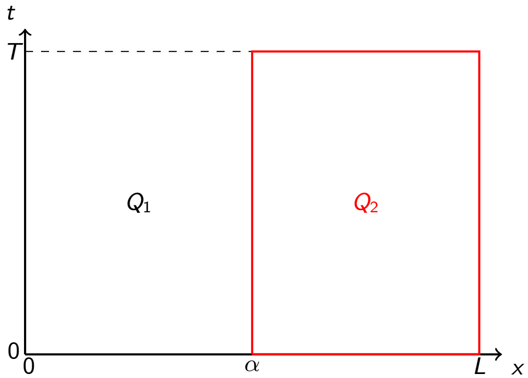
$$\begin{aligned} \partial_t y - \Delta y &= \nu^{-1}\lambda, & \partial_t \lambda + \Delta \lambda &= y - \hat{y}, \\ y(0,t) &= 0, & \lambda(0,t) &= 0, \\ y(L,t) &= 0, & \lambda(L,t) &= 0, \\ y(x,0) &= y_0(x), & \lambda(x,T) + \gamma y(x,T) &= \gamma \hat{y}(x,T). \end{aligned}$$



$Q_1 = (0, \alpha) \times (0, T)$,

$$\begin{aligned} \partial_t y_1^{k+1} - \Delta y_1^{k+1} &= \nu^{-1}\lambda_1^{k+1}, & \partial_t \lambda_1^{k+1} + \Delta \lambda_1^{k+1} &= y_1^{k+1} - \hat{y}_1, \\ y_1^{k+1}(0,t) &= 0, & \lambda_1^{k+1}(0,t) &= 0, \\ y_1^{k+1}(\alpha,t) &= y_2^{k}(\alpha,t), & \lambda_1^{k+1}(\alpha,t) &= \lambda_2^{k}(\alpha,t), \\ y_1^{k+1}(x,0) &= y_{1,0}(x), & \lambda_1^{k+1}(x,T) + \gamma y_1^{k+1}(x,T) &= \gamma \hat{y}_1(x,T). \end{aligned}$$

Dirichlet-Neumann Waveform Relaxation



$Q_2 = (\alpha, 1) \times (0, T)$,

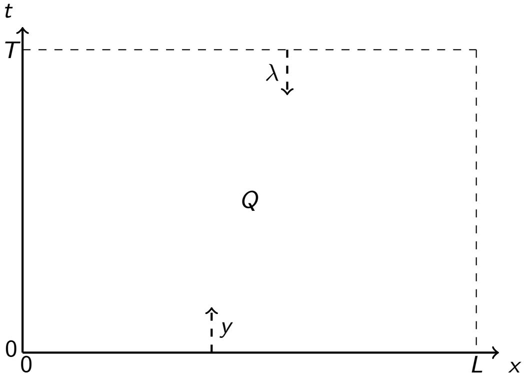
$$\partial_t y_2^{k+1} - \Delta y_2^{k+1} = \nu^{-1} \lambda_2^{k+1}, \qquad \partial_t \lambda_2^{k+1} + \Delta \lambda_2^{k+1} = y_2^{k+1} - \hat{y}_2,$$

$$\partial_x y_2^{k+1}(\alpha, t) = \partial_x y_1^{k+1}(\alpha, t), \qquad \partial_x \lambda_2^{k+1}(\alpha, t) = \partial_x \lambda_1^{k+1}(\alpha, t),$$

$$y_2^{k+1}(L, t) = 0, \qquad \lambda_2^{k+1}(L, t) = 0,$$

$$y_2^{k+1}(x, 0) = y_{2,0}(x), \qquad \lambda_2^{k+1}(x, T) + \gamma y_2^{k+1}(x, T) = \gamma \hat{y}_2(x, T).$$

Time domain decomposition



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$$\begin{aligned} \partial_t y - \Delta y &= \nu^{-1}\lambda, & \partial_t \lambda + \Delta \lambda &= y - \hat{y}, \\ y(0,t) &= 0, & \lambda(0,t) &= 0, \\ y(L,t) &= 0, & \lambda(L,t) &= 0, \\ y(x,0) &= y_0(x), & \lambda(x,T) + \gamma y(x,T) &= \gamma \hat{y}(x,T). \end{aligned}$$

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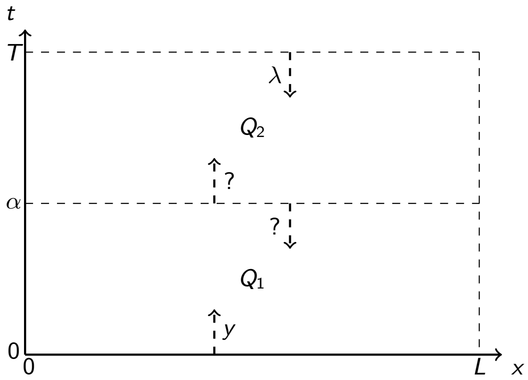

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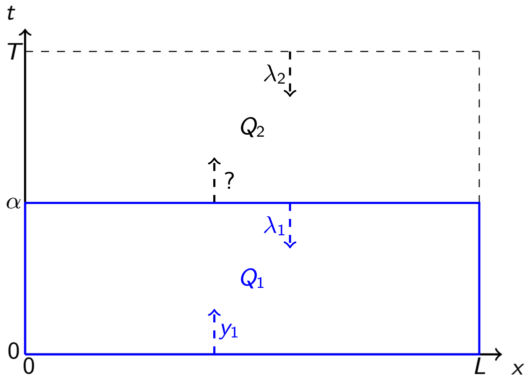
Time domain decomposition



$Q_1 = (0, L) \times (0, \alpha)$,

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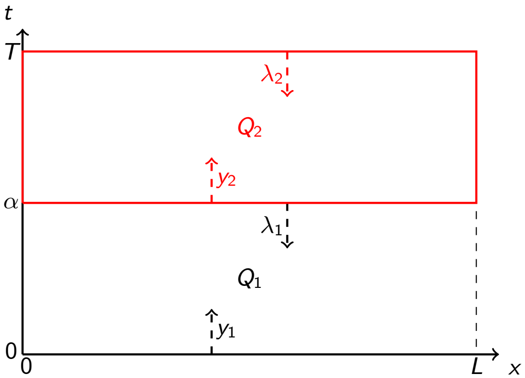

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Reduced optimality system:

$$\begin{cases} \partial_t \begin{pmatrix} y \\ \lambda \end{pmatrix} + \begin{pmatrix} -\Delta y - \nu^{-1}\lambda \\ -y + \Delta\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{y} \end{pmatrix}, \\ \hfill y(\cdot, 0) = y_0(\cdot), \\ \hfill \lambda(\cdot, T) + \gamma y(\cdot, T) = \gamma\hat{y}(\cdot, T). \end{cases}$$

Ex: finite difference $-\Delta \approx A$

$$\begin{cases} \begin{pmatrix} \dot{Y} \\ \dot{\Lambda} \end{pmatrix} + \begin{pmatrix} A & -\nu^{-1} I \\ -I & -A \end{pmatrix} \begin{pmatrix} Y \\ \Lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{Y} \end{pmatrix} \text{ in } (0, T), \\ Y(0) = Y_0, \\ \Lambda(T) + \gamma Y(T) = \gamma \hat{Y}(T). \end{cases}$$

Semi-discretization

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$A = PDP^{-1}$ and $D = \operatorname{diag}(d_1, \dots, d_n)$,

$$\begin{cases} \begin{pmatrix} \dot{z}_i \\ \dot{\mu}_i \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_i \\ \mu_i \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{z}_i \end{pmatrix} \text{ in } (0, T), \\ z_i(0) = z_{0,i}, \\ \mu_i(T) + \gamma z_i(T) = \gamma \hat{z}_i(T), \end{cases}$$

with $z = P^{-1}Y$, $\hat{z} = P^{-1}\hat{Y}$ and $\mu = P^{-1}\Lambda$. n **independent** 2×2 **systems**.

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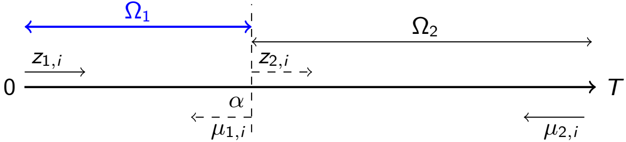
with $z = P^{-1}Y$, $\hat{z} = P^{-1}\hat{Y}$ and $\mu = P^{-1}\Lambda$. n **independent** 2×2 **systems**.
Second-order ODE:

$$\begin{cases} \ddot{z}_i - \sigma_i^2 z_i = -\nu^{-1}\hat{z}_i \text{ in } (0, T), \\ z_i(0) = z_{0,i}, \\ \dot{z}_i(T) + \omega_i z_i(T) = \nu^{-1}\gamma \hat{z}_i(T), \end{cases} \text{ or } \begin{cases} \ddot{\mu}_i - \sigma_i^2 \mu_i = -(\dot{\hat{z}}_i + d_i \hat{z}_i) \text{ in } (0, T), \\ \mu_i(0) - d_i \mu_i(0) = z_{0,i} - \hat{z}_i(0), \\ \gamma \mu_i(T) + \beta_i \mu_i(T) = 0, \end{cases}$$

with $\sigma_i := \sqrt{d_i^2 + \nu^{-1}}$, $\omega_i := \nu^{-1}\gamma + d_i$ and $\beta_i := 1 - \gamma d_i$.

Transformation



Dirichlet:

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^k \\ \dot{\mu}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^k \\ \mu_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^k(0) = 0, \\ \mu_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}, \end{cases}$$

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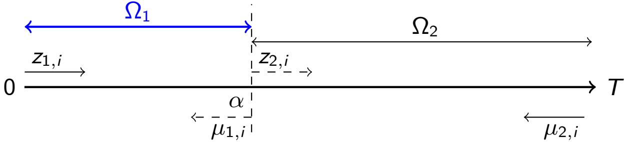

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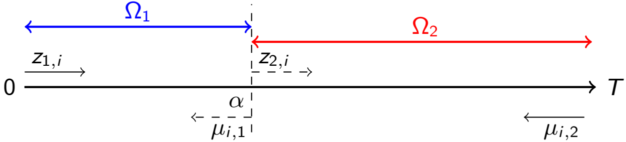
Update:

$$f_{\alpha,i}^k := (1-\theta) f_{\alpha,i}^{k-1} + \theta \mu_{2,i}^k(\alpha), \quad \theta \in (0,1).$$



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Neumann:

$$\begin{cases} \begin{pmatrix} \dot{z}_{2,i}^k \\ \dot{\mu}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^k \\ \mu_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}_{2,i}^k(\alpha) = \dot{z}_{1,i}^k(\alpha), \\ \mu_{2,i}^k(T) + \gamma z_{2,i}^k(T) = 0, \end{cases}$$

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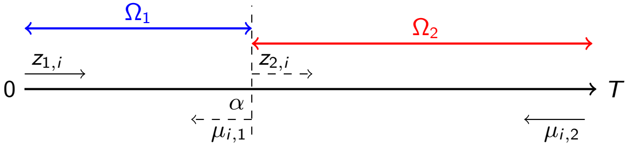
$$f_{\alpha,i}^k := (1-\theta) f_{\alpha,i}^{k-1} + \theta \mu_{2,i}^k(\alpha), \quad \theta \in (0,1).$$



Dirichlet:

$$\begin{cases} \ddot{z}_{1,i}^k - \sigma_i^2 z_{1,i}^k = 0 \text{ in } \Omega_1, \\ z_{1,i}^k(0) = 0, \\ \dot{z}_{1,i}^k(\alpha) + d_i z_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}, \end{cases}$$

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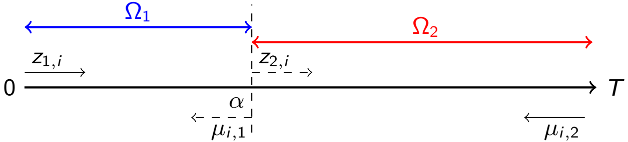
Update:

$$f_{\alpha,i}^k = (1-\theta) f_{\alpha,i}^{k-1} + \theta\big(\dot{z}_{2,i}^k(\alpha) + d_i z_{2,i}^k(\alpha)\big).$$



Dirichlet:

$$\begin{cases} \ddot{\mu}_{1,i}^k - \sigma_i^2 \mu_{1,i}^k = 0 \text{ in } \Omega_1, \\ \dot{\mu}_i(0) - d_i \mu_i(0) = 0, \\ \mu_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}, \end{cases}$$

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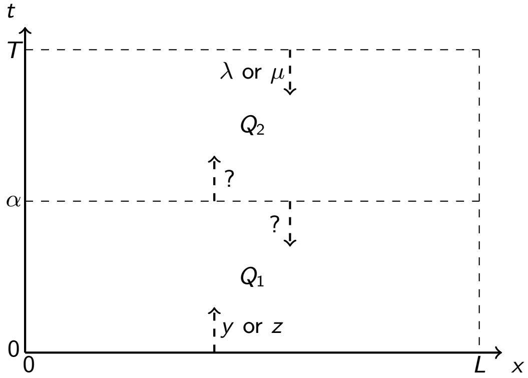
$$f_{\alpha,i}^k := (1-\theta) f_{\alpha,i}^{k-1} + \theta \mu_{2,i}^k(\alpha), \quad \theta \in (0,1).$$

Observations:

- three systems are equivalent,
- same convergence using z or μ,
- not anymore a "DN" algorithm,
- forward-backward structure less important.



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$$
\begin{aligned}
\partial_t y - \Delta y &= \nu^{-1}\lambda, & \partial_t \lambda + \Delta \lambda &= y - \hat{y},\\
y(0, t) &= 0, & \lambda(0, t) &= 0,\\
y(L, t) &= 0, & \lambda(L, t) &= 0,\\
y(x, 0) &= y_0(x), & \lambda(x, T) + \gamma y(x, T) &= \gamma \hat{y}(x, T).
\end{aligned}
$$

Variants of DN and ND algorithms

Category	Ω_1	Ω_2	type
(z_i, μ_i)	μ_i	$\dot{z}_i$	(DN)
	$\dot{z}_i + d_i z_i$	$\dot{z}_i$	(RN)
	$\dot{\mu}_i$	z_i	(ND)
	$\ddot{z}_i + d_i \dot{z}_i$	z_i	(RD)
z_i	z_i	$\dot{z}_i$	(DN)
	z_i	$\dot{z}_i$	(DN)
	$\dot{z}_i$	z_i	(ND)
	$\dot{z}_i$	z_i	(ND)
μ_i	μ_i	$\dot{\mu}_i$	(DN)
	$\dot{z}_i + d_i z_i$	$\ddot{z}_i + d_i \dot{z}_i$	(RR)
	$\dot{\mu}_i$	μ_i	(ND)
	$\ddot{z}_i + d_i \dot{z}_i$	$\dot{z}_i + d_i z_i$	(RR)

Comparaison of two DN variants

Natural Dirichlet-Neumann (DN_1):

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^k \\ \dot{\mu}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^k \\ \mu_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^k(0) = 0, \\ \mu_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}, \end{cases} \quad \begin{cases} \begin{pmatrix} \dot{z}_{2,i}^k \\ \dot{\mu}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^k \\ \mu_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}_{2,i}^k(\alpha) = \dot{z}_{1,i}^k(\alpha), \\ \mu_{2,i}^k(T) + \gamma z_{2,i}^k(T) = 0, \end{cases}$$

$$f_{\alpha,i}^k := (1-\theta) f_{\alpha,i}^{k-1} + \theta \mu_{2,i}^k(\alpha).$$

Dirichlet-Neumann on both level (DN_2):

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^k \\ \dot{\mu}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^k \\ \mu_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^k(0) = 0, \\ z_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}, \end{cases} \quad \begin{cases} \begin{pmatrix} \dot{z}_{2,i}^k \\ \dot{\mu}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^k \\ \mu_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}_{2,i}^k(\alpha) = \dot{z}_{1,i}^k(\alpha), \\ \mu_{2,i}^k(T) + \gamma z_{2,i}^k(T) = 0, \end{cases}$$

$$f_{\alpha,i}^k := (1-\theta) f_{\alpha,i}^{k-1} + \theta z_{2,i}^k(\alpha).$$

Forward-backward can always be recovered !

$$z_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1} \Rightarrow \dot{\mu}_{1,i}^k(\alpha) - d_i \mu_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}.$$

Comparaison of two DN variants

Natural Dirichlet-Neumann (DN_1):

$$\begin{cases} \ddot{z}^k_{1,i} - \sigma_i^2 z^k_{1,i} = 0 \text{ in } \Omega_1, \\ z^k_{1,i}(0) = 0, \\ \dot{z}^k_{1,i}(\alpha) + d_i z^k_{1,i}(\alpha) = f^{k-1}_{\alpha,i}, \end{cases} \quad \begin{cases} \ddot{z}^k_{2,i} - \sigma_i^2 z^k_{2,i} = 0 \text{ in } \Omega_2, \\ \dot{z}^k_{2,i}(\alpha) = \dot{z}^k_{1,i}(\alpha), \\ \dot{z}^k_{2,i}(T) + \omega_i z^k_{2,i}(T) = 0, \end{cases}$$

$$f^k_{\alpha,i} := (1-\theta) f^{k-1}_{\alpha,i} + \theta\big(\dot{z}^k_{2,i}(\alpha) + d_i z^k_{2,i}(\alpha)\big).$$

Dirichlet-Neumann on both level (DN_2):

$$\begin{cases} \ddot{z}^k_{1,i} - \sigma_i^2 z^k_{1,i} = 0 \text{ in } \Omega_1, \\ z^k_{1,i}(0) = 0, \\ z^k_{1,i}(\alpha) = f^{k-1}_{\alpha,i}, \end{cases} \quad \begin{cases} \ddot{z}^k_{2,i} - \sigma_i^2 z^k_{2,i} = 0 \text{ in } \Omega_2, \\ \dot{z}^k_{2,i}(\alpha) = \dot{z}^k_{1,i}(\alpha), \\ \dot{z}^k_{2,i}(T) + \omega_i z^k_{2,i}(T) = 0, \end{cases}$$

$$f^k_{\alpha,i} := (1-\theta) f^{k-1}_{\alpha,i} + \theta z^k_{2,i}(\alpha).$$

Dirichlet-Neumann convergence analysis

Solve the problem and find

$$f_{\alpha,i}^{k} := \rho \, f_{\alpha,i}^{k-1}.$$

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Convergence factor with analytical form

$$\rho_{\mathrm{DN}_1} := \max_{d_i \in \lambda(A)} \left| 1 - \theta\left(1 - \nu^{-1} \frac{\sigma_i \gamma + \beta_i \tanh(b_i)}{\big(\sigma_i + d_i \tanh(a_i)\big)\big(\omega_i + \sigma_i \tanh(b_i)\big)}\right)\right|,$$

$$\rho_{\mathrm{ND}_1} := \max_{d_i \in \lambda(A)} \left| 1 - \theta\left(1 - \nu^{-1} \frac{\sigma_i \gamma + \beta_i \coth(b_i)}{\big(\sigma_i + d_i \coth(a_i)\big)\big(\omega_i + \sigma_i \coth(b_i)\big)}\right)\right|.$$

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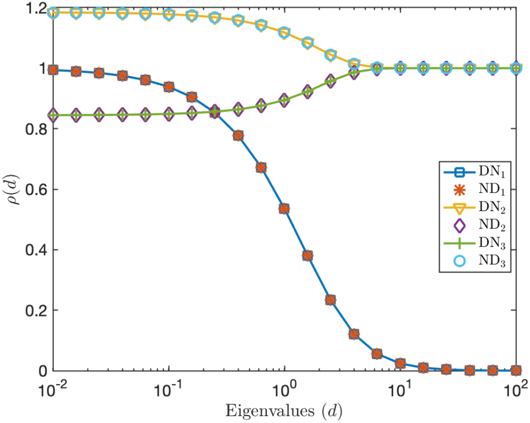
"Optimal" relaxation parameter with equioscillation

$$\theta^*_{\mathrm{DN}_2} = \frac{2}{3+\coth(\sqrt{\nu^{-1}}\alpha)\frac{\coth\left(\sqrt{\nu^{-1}}(T-\alpha)\right)+\gamma\sqrt{\nu^{-1}}}{1+\gamma\sqrt{\nu^{-1}}\coth\left(\sqrt{\nu^{-1}}(T-\alpha)\right)}},$$

$$\theta^*_{\mathrm{ND}_2} = \frac{2}{3+\tanh(\sqrt{\nu^{-1}}\alpha)\frac{\tanh\left(\sqrt{\nu^{-1}}(T-\alpha)\right)+\gamma\sqrt{\nu^{-1}}}{1+\gamma\sqrt{\nu^{-1}}\tanh\left(\sqrt{\nu^{-1}}(T-\alpha)\right)}},$$

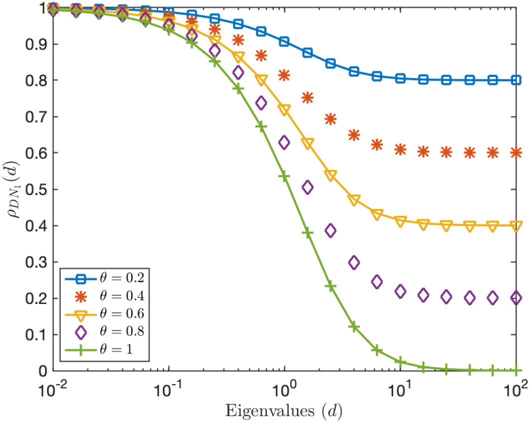
$\theta^*_{\mathrm{DN}_2} = \theta^*_{\mathrm{ND}_3}$ and $\theta^*_{\mathrm{ND}_2} = \theta^*_{\mathrm{DN}_3}$.

$\nu = 0.1$, $\gamma = 0$, $\alpha = \frac{T}{2}$ and $\theta = 1$.

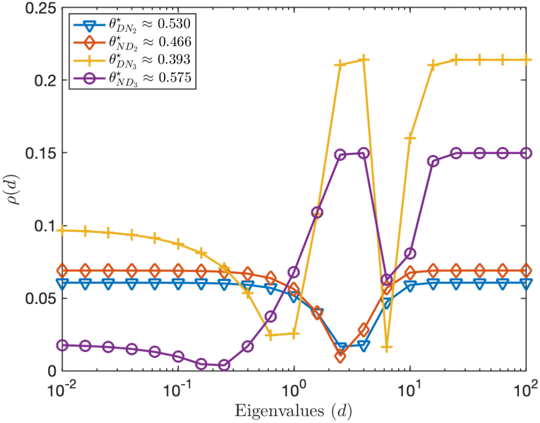


Numerical experiments

$\nu = 0.1$, $\gamma = 0$ and $\alpha = \frac{T}{2}$.

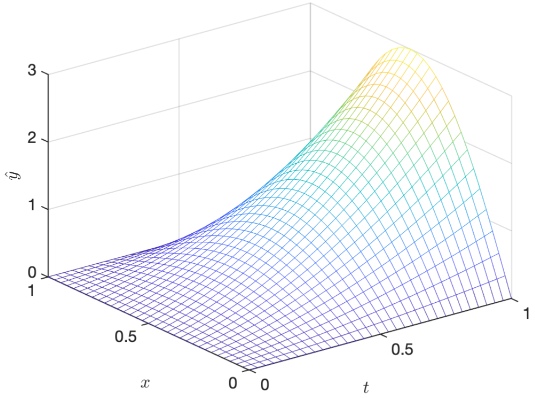


$\nu = 0.1$, $\gamma = 10$ and $\alpha = \frac{7}{10}T$.



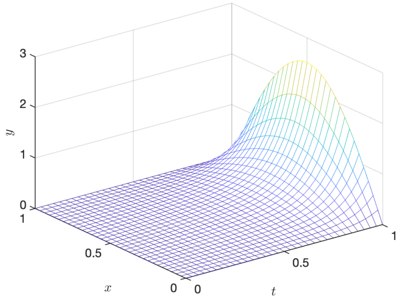
$\theta^*_{\mathrm{DN}_2} = \theta^\star_{\mathrm{DN}_2} \neq \theta^\star_{\mathrm{ND}_3}$ and $\theta^*_{\mathrm{ND}_2} = \theta^\star_{\mathrm{ND}_2} \neq \theta^\star_{\mathrm{DN}_3}$.

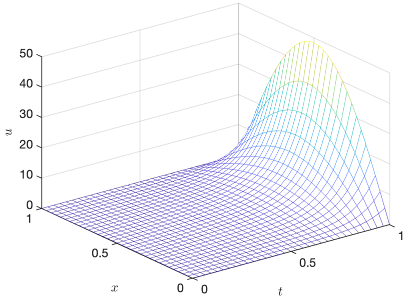
$\nu = 0.1$, $\gamma = 10$ and $T = 1$, $\hat{y}(t, x) = \sin(\pi x)(2t^2 + t)$.



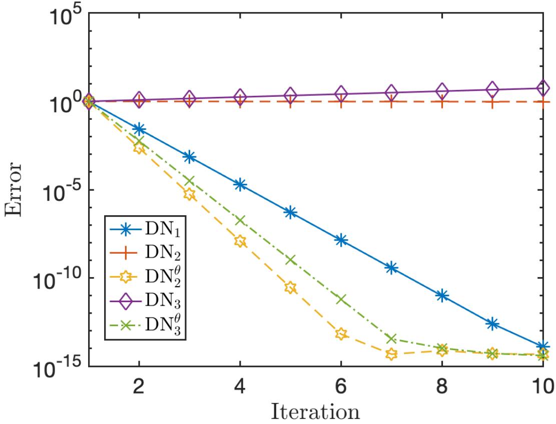
Numerical experiments

Crank-Nicolson $h_t = h_x = \frac{1}{32}$.

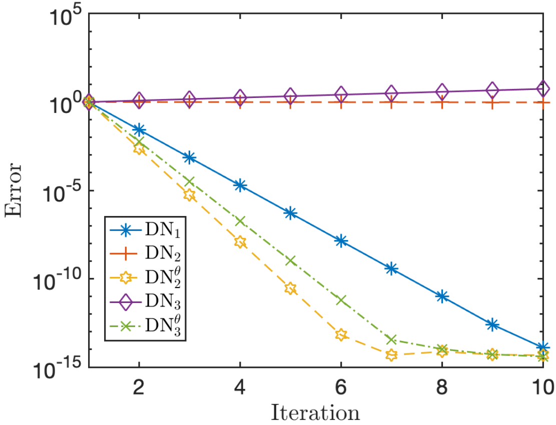




$\alpha = \frac{7}{10}T$.



$\alpha = \frac{7}{10}T.$



Thanks for your attention !