Parallel-in-Time algorithm for parabolic optimal control problems

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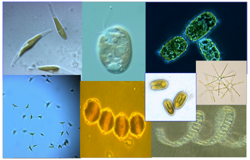
Matematikcentrums, Lunds Universitet

Scientific Computing and Differential Equations 2026

MS54: New iterative methods and their analysis for linear and non-linear problems

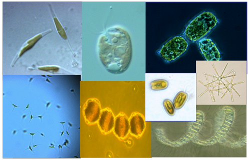
The University of Edinburgh, July 2, 2026

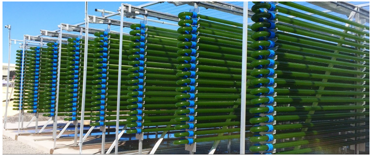
Application: biomass production



Microalgae: photosynthetic micro-organisms 2-50 μm in acquatic environment

Application: biomass production





Microalgae: photosynthetic micro-organisms 2-50 μm in acquatic environment

Photobioreactors:

- technology: chemostats, raceway, tubular, etc
- position: indoor/outdoor
- scale: lab/industrial
- purpose: wastewater treatment, biofuel production, etc

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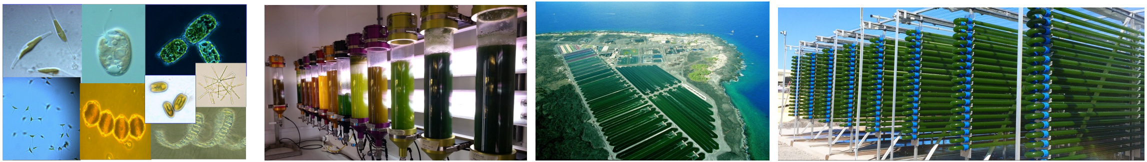

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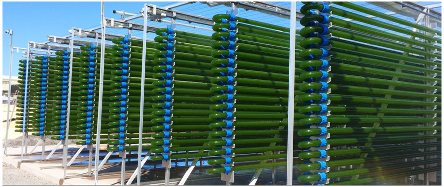
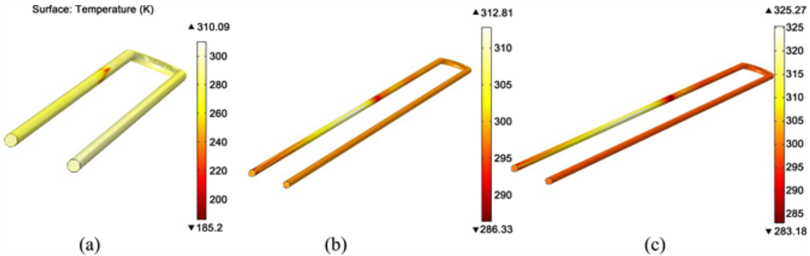
Challenges:

- both physical and biological behavior
- coupled models
- different timescales
- macro/micro levels
- uncertainties: environment, measurement, etc

Example: tubular photobioreactor





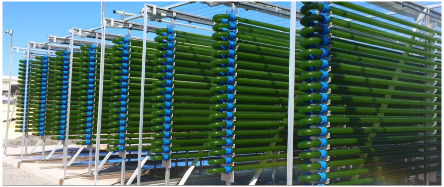
Example: tubular photobioreactor

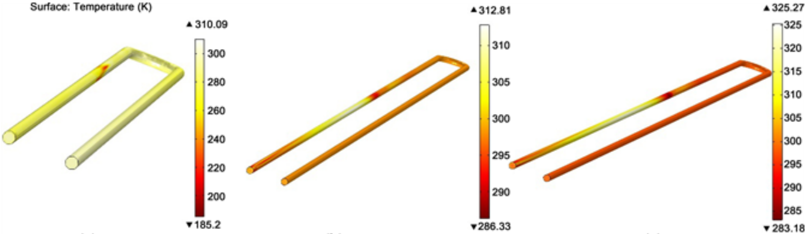
Minimize temperature distribution w.r.t. target $\hat{y}$

$$J(y,u) = \frac{1}{2}\|y-\hat{y}\|^2_{L^2(Q)} + \frac{\nu}{2}\|u\|^2_{L^2(Q)}$$

Governing equation: heat equation

$$\partial_t y - \kappa \Delta y = u$$





(a) (b) (c)

Optimal control problems

Idea: Find the optimal strategy to pilot the system to behave as we wish

Ingredients:

- A cost functional: $J(y,u) = \frac{1}{2}\|y-\hat{y}\|^2_{L^2(Q)} + \frac{\nu}{2}\|u\|^2_{L^2(Q)}$
- A system governed by an ODE/PDE: $\partial_t y - \Delta y = u$ in $Q := \Omega \times (0,T)$
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Approach: Lagrange multiplier p

$$\mathcal{L}(y, p, u) = J(y, u) + \langle p, \partial_t y - \Delta y - u \rangle$$

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Approach: Lagrange multiplier p

$$\mathcal{L}(y,p,u) = J(y,u) + \langle p, \partial_t y - \Delta y - u \rangle$$

Derive first-order optimality system: $\partial_p \mathcal{L}[\delta p] = 0$, $\partial_y \mathcal{L}[\delta y] = 0$, $\partial_u \mathcal{L}[\delta u] = 0$

$$\begin{aligned} \partial_t y - \Delta y &= u && \text{in } Q & \qquad \partial_t p + \Delta p &= y - \hat{y} && \text{in } Q \\ y &= 0 && \text{on } \Sigma & p &= 0 && \text{on } \Sigma \\ y &= y_0 && \text{on } \Sigma_0 & p &= 0 && \text{on } \Sigma_T \end{aligned}$$

$$\nu u - p = 0 \quad \text{in } Q$$

Summary of PinT for OCPs

- Accelerator for gradient or Newton iterations: Parareal, PFASST, MGRIT, ...
  (inner solver, no intrusive, natural fit for legacy solvers, easy to generalize, ...)

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- Parallel-then-optimize: decompose problem first, augmented Lagrangian
  (optimization-aware parallelism, local solvers, break long time horizon, ...)

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Parallel-in-Time methods for PDE-constrained optimization, Gander, L., Wu, In preparation

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Apply domain decomposition

Reduced optimality system (forward-backward)

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\partial_t y - \Delta y &= \nu^{-1} p && \text{in } Q \qquad & \partial_t p + \Delta p &= y - \hat{y} && \text{in } Q\\
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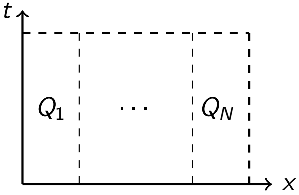
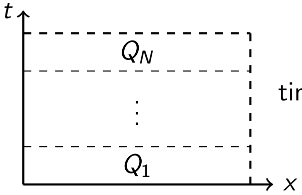
Two ways of decomposition

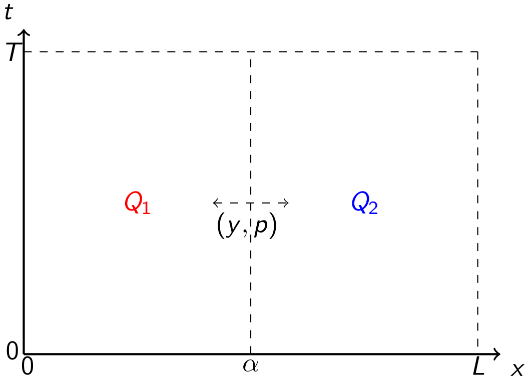
space decomposition





time decomposition

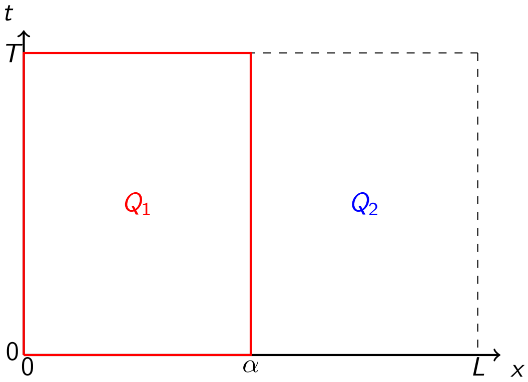
Space decomposition (Waveform method)



Subdomains: $Q_1 = (0, \alpha) \times (0, T)$ and $Q_2 = (\alpha, L) \times (0, T)$

$$
\begin{aligned}
\partial_t y - \partial_{xx} y &= \nu^{-1} p & \qquad \partial_t p + \partial_{xx} p &= y - \hat{y} \\
y(0, t) &= 0 & p(0, t) &= 0 \\
y(L, t) &= 0 & p(L, t) &= 0 \\
y(x, 0) &= y_0(x) & p(x, T) &= 0
\end{aligned}
$$

Alternating Schwarz waveform



In $Q_1 = (0,\alpha) \times (0,T)$

$$\partial_t y_1^\ell - \partial_{xx} y_1^\ell = \nu^{-1} p_1^\ell \qquad \partial_t p_1^\ell + \partial_{xx} p_1^\ell = y_1^\ell - \hat{y}_1$$

$$y_1^\ell(0,t) = 0 \qquad p_1^\ell(0,t) = 0$$

$$y_1^\ell(x,0) = y_{1,0}(x) \qquad p_1^\ell(x,T) = 0$$

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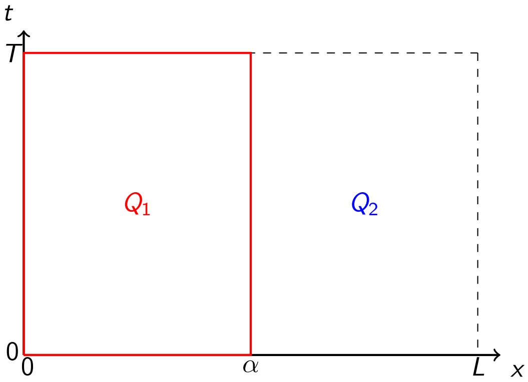

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y_1^\ell(\alpha,t) &= y_2^{\ell-1}(\alpha,t) & p_1^\ell(\alpha,t) &= p_2^{\ell-1}(\alpha,t) \\
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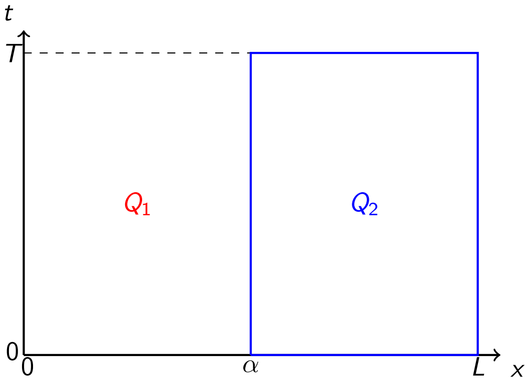

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y_1^\ell(x,0) &= y_{1,0}(x) & p_1^\ell(x,T) &= 0
\end{aligned}
$$

In $Q_2 = (\alpha,L)\times(0,T)$

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\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2\\
y_2^\ell(L,t) &= 0 & p_2^\ell(L,t) &= 0\\
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\end{aligned}
$$

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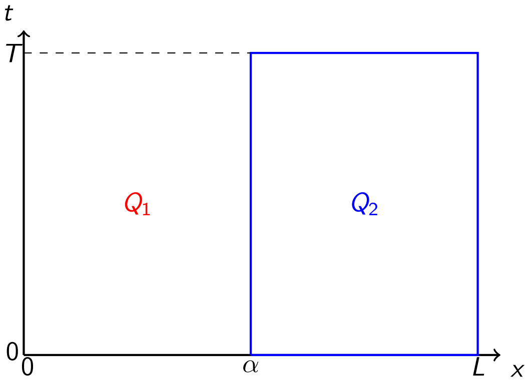

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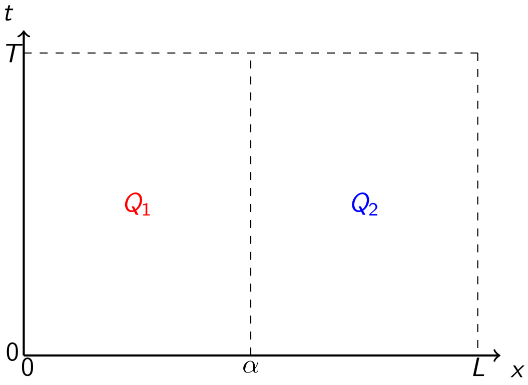

Does not converge without overlap !

In $Q_1 = (0,\alpha) \times (0,T)$

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In $Q_2 = (\alpha,L) \times (0,T)$

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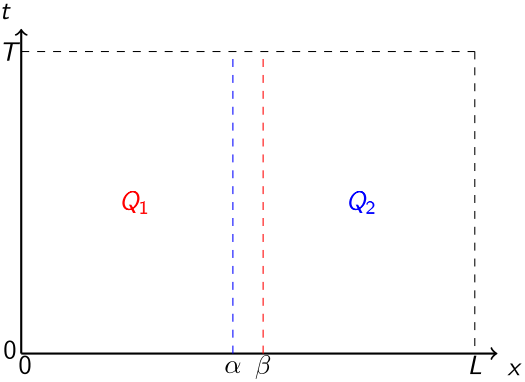

Now converges with overlap !

In $Q_1 = (0,\beta)\times(0,T)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1\\
y_1^\ell(0,t) &= 0 & p_1^\ell(0,t) &= 0\\
y_1^\ell(\beta,t) &= y_2^{\ell-1}(\beta,t) & p_1^\ell(\beta,t) &= p_2^{\ell-1}(\beta,t)\\
y_1^\ell(x,0) &= y_{1,0}(x) & p_1^\ell(x,T) &= 0
\end{aligned}
$$

In $Q_2 = (\alpha,L)\times(0,T)$

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\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2\\
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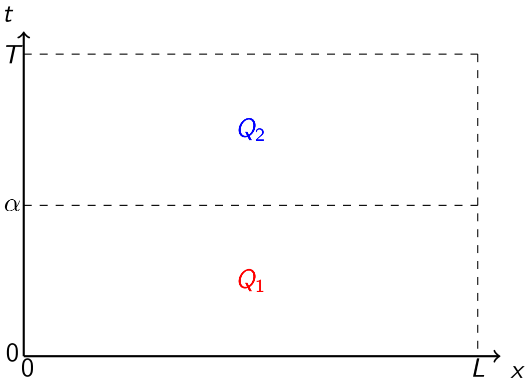
$$

Time decomposition



Subdomains: $Q_1 = (0, L) \times (0, \alpha)$ and $Q_2 = (0, L) \times (\alpha, T)$

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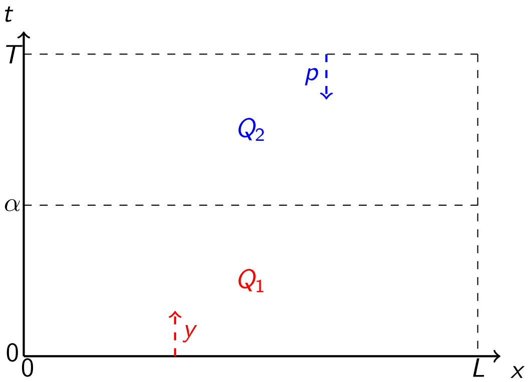

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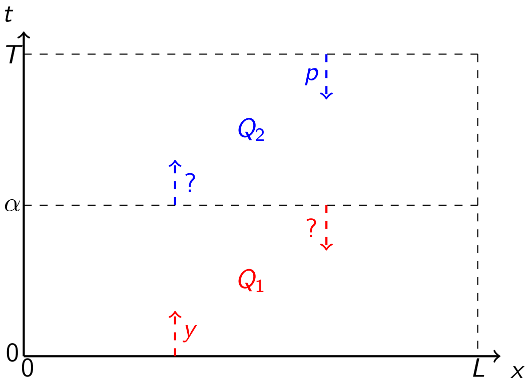

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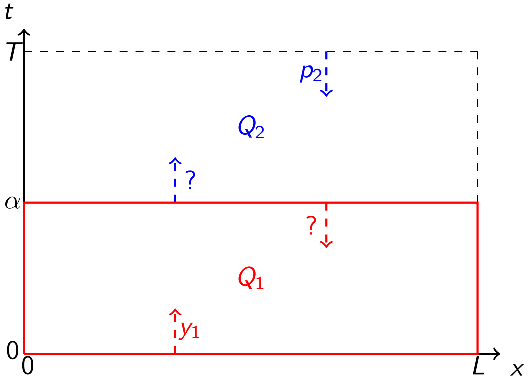
Alternating Schwarz in time



In $Q_1 = (0, L) \times (0, \alpha)$

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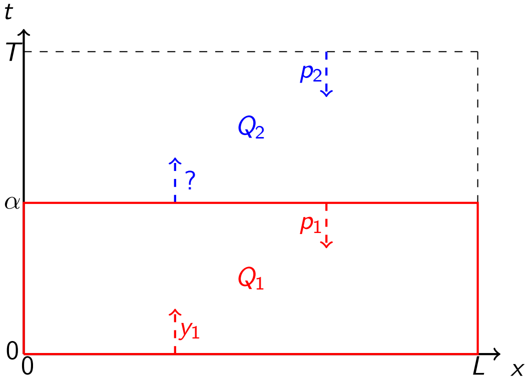

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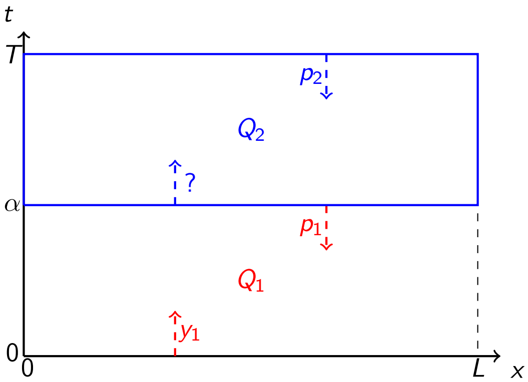

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In $Q_2 = (0, L) \times (\alpha, T)$

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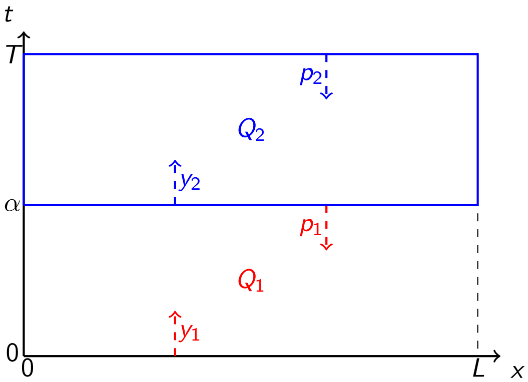

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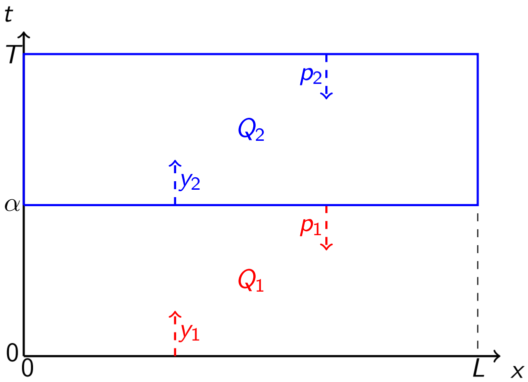
$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\
y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\
y_1^\ell(x, 0) &= y_0(x) & p_1^\ell(x, \alpha) &= p_2^{\ell-1}(x, \alpha)
\end{aligned}
$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$
\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\
y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\
y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\
y_2^\ell(x, \alpha) &= y_1^\ell(x, \alpha) & p_2^\ell(x, T) &= 0
\end{aligned}
$$

Alternating Schwarz in time



How about the convergence ?

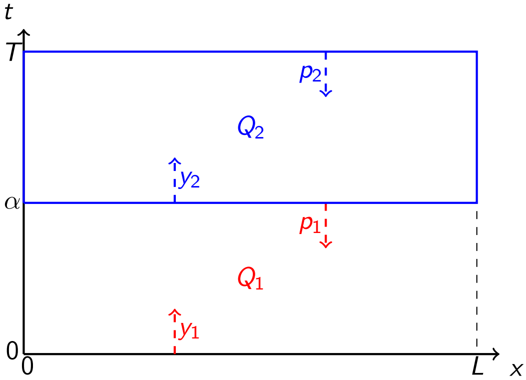
In $Q_1 = (0, L) \times (0, \alpha)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \quad \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\
y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\
y_1^\ell(x, 0) &= y_0(x) & p_1^\ell(x, \alpha) &= p_2^{\ell-1}(x, \alpha)
\end{aligned}
$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$
\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \quad \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\
y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\
y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\
y_2^\ell(x, \alpha) &= y_1^\ell(x, \alpha) & p_2^\ell(x, T) &= 0
\end{aligned}
$$

Alternating Schwarz in time



How about the convergence ?
It **does converge** without overlap !

In $Q_1 = (0, L) \times (0, \alpha)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\
y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\
y_1^\ell(x, 0) &= y_0(x) & p_1^\ell(x, \alpha) &= p_2^{\ell-1}(x, \alpha)
\end{aligned}
$$

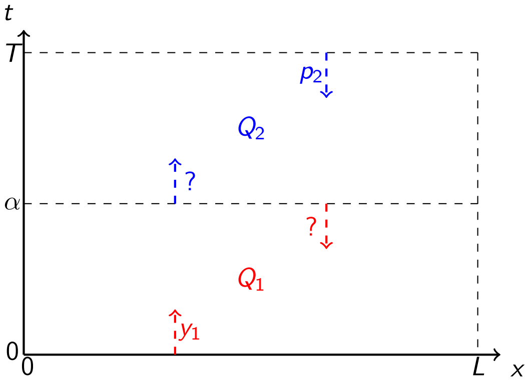
In $Q_2 = (0, L) \times (\alpha, T)$

$$
\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\
y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\
y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\
y_2^\ell(x, \alpha) &= y_1^\ell(x, \alpha) & p_2^\ell(x, T) &= 0
\end{aligned}
$$

Alternating Schwarz in time - Intuition

Relations

$$p_j^\ell = \nu(\partial_t y_j^\ell - \partial_{xx} y_j^\ell)$$
$$y_j^\ell = \partial_t p_j^\ell + \partial_{xx} p_j^\ell + \hat{y}_j$$



In $Q_1 = (0, L) \times (0, \alpha)$

$$\partial_t y_1^\ell - \partial_{xx} y_1^\ell = \nu^{-1} p_1^\ell \quad \partial_t p_1^\ell + \partial_{xx} p_1^\ell = y_1^\ell - \hat{y}_1$$

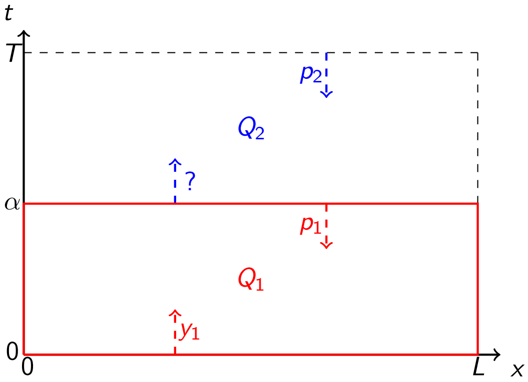
In $Q_2 = (0, L) \times (\alpha, T)$

$$\partial_t y_2^\ell - \partial_{xx} y_2^\ell = \nu^{-1} p_2^\ell \quad \partial_t p_2^\ell + \partial_{xx} p_2^\ell = y_2^\ell - \hat{y}_2$$

Alternating Schwarz in time - Intuition

Relations

$$p_j^\ell = \nu(\partial_t y_j^\ell - \partial_{xx} y_j^\ell)$$



In $Q_1 = (0, L) \times (0, \alpha)$

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0,t) &= 0 & p_1^\ell(0,t) &= 0 \\ y_1^\ell(L,t) &= 0 & p_1^\ell(L,t) &= 0 \\ y_1^\ell(x,0) &= y_0(x) & p_1^\ell(x,\alpha) &= p_2^{\ell-1}(x,\alpha) \end{aligned}$$

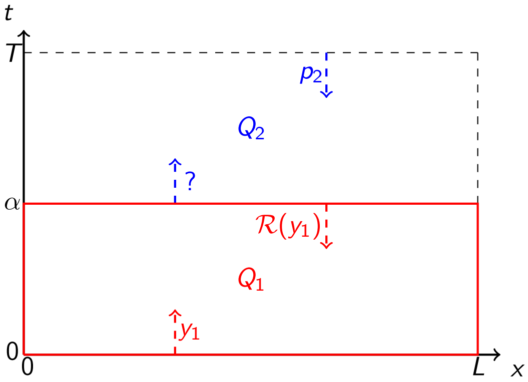
In $Q_2 = (0, L) \times (\alpha, T)$

$$\partial_t y_2^\ell - \partial_{xx} y_2^\ell = \nu^{-1} p_2^\ell \quad \partial_t p_2^\ell + \partial_{xx} p_2^\ell = y_2^\ell - \hat{y}_2$$

Alternating Schwarz in time - Intuition

Relations

$$p_j^\ell = \nu(\partial_t y_j^\ell - \partial_{xx} y_j^\ell)$$



In $Q_1 = (0, L) \times (0, \alpha)$

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\ y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\ y_1^\ell(x, 0) &= y_0(x) \end{aligned}$$

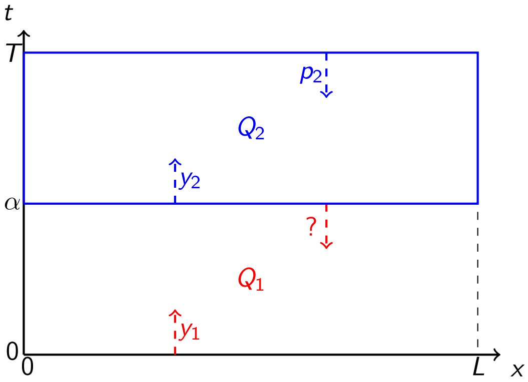
$$(\partial_t - \partial_{xx}) y_1^\ell(x, \alpha) = (\partial_t - \partial_{xx}) y_2^{\ell-1}(x, \alpha)$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\partial_t y_2^\ell - \partial_{xx} y_2^\ell = \nu^{-1} p_2^\ell \quad \partial_t p_2^\ell + \partial_{xx} p_2^\ell = y_2^\ell - \hat{y}_2$$

Relations

$$y_j^\ell = \partial_t p_j^\ell + \partial_{xx} p_j^\ell + \hat{y}_j$$



In $Q_1 = (0, L) \times (0, \alpha)$

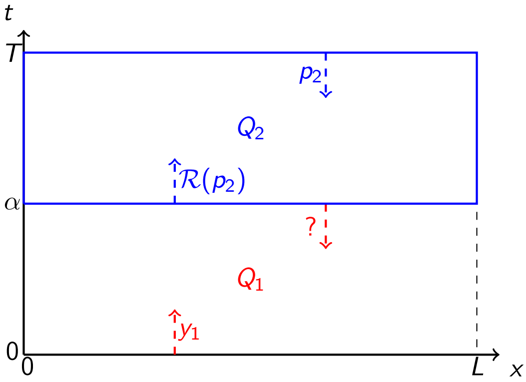
$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0,t) &= 0 & p_1^\ell(0,t) &= 0 \\ y_1^\ell(L,t) &= 0 & p_1^\ell(L,t) &= 0 \\ y_1^\ell(x,0) &= y_0(x) \end{aligned}$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0,t) &= 0 & p_2^\ell(0,t) &= 0 \\ y_2^\ell(L,t) &= 0 & p_2^\ell(L,t) &= 0 \\ y_2^\ell(x,\alpha) &= y_1^\ell(x,\alpha) & p_2^\ell(x,T) &= 0 \end{aligned}$$

Relations

$$y_j^\ell = \partial_t p_j^\ell + \partial_{xx} p_j^\ell + \hat{y}_j$$



In $Q_1 = (0, L) \times (0, \alpha)$

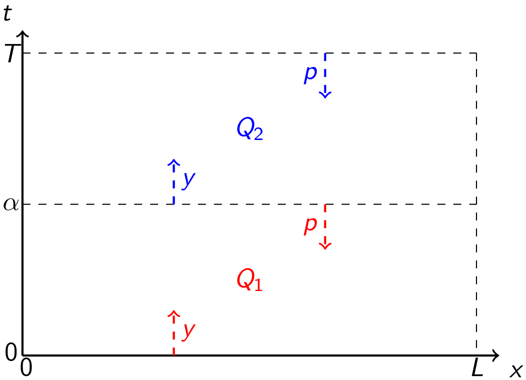
$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\ y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\ y_1^\ell(x, 0) &= y_0(x) \end{aligned}$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\ y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\ & & p_2^\ell(x, T) &= 0 \end{aligned}$$

$$(\partial_t + \partial_{xx}) p_2^\ell(x, \alpha) = (\partial_t + \partial_{xx}) p_1^\ell(x, \alpha)$$

Alternating Schwarz in time - Observation



- **Dirichlet** condition transform to **Robin** type condition !

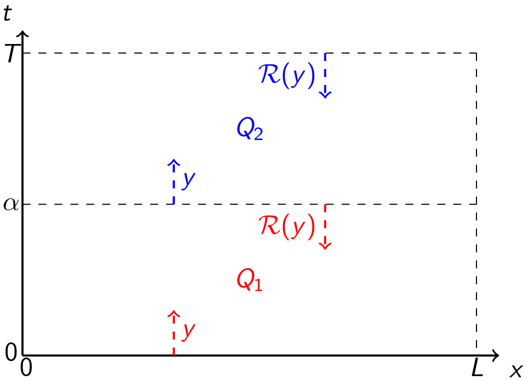
In $Q_1 = (0, L) \times (0, \alpha)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\
y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\
y_1^\ell(x, 0) &= y_0(x) & p_1^\ell(x, \alpha) &= p_2^{\ell-1}(x, \alpha)
\end{aligned}
$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$
\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\
y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\
y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\
y_2^\ell(x, \alpha) &= y_1^\ell(x, \alpha) & p_2^\ell(x, T) &= 0
\end{aligned}
$$

Alternating Schwarz in time - Observation



In $Q_1 = (0, L) \times (0, \alpha)$

$$
\begin{aligned}
-\partial_{tt} y_1^\ell + \Delta^2 y_1^\ell + \nu^{-1} y_1^\ell &= \nu^{-1} \hat{y}_1 \\
y_1^\ell(x, 0) &= y_0(x) \\
(\partial_t - \partial_{xx}) y_1^\ell(x, \alpha) &= (\partial_t - \partial_{xx}) y_2^{\ell-1}(x, \alpha) \\
&+ \text{ B.C.}
\end{aligned}
$$

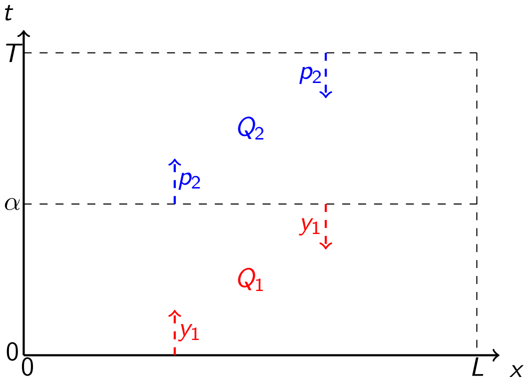
In $Q_2 = (0, L) \times (\alpha, T)$

$$
\begin{aligned}
-\partial_{tt} y_2^\ell + \Delta^2 y_2^\ell + \nu^{-1} y_2^\ell &= \nu^{-1} \hat{y}_2 \\
y_2^\ell(x, \alpha) &= y_1^\ell(x, \alpha) \\
(\partial_t - \partial_{xx}) y_2^{\ell-1}(x, T) &= 0 \\
&+ \text{ B.C.}
\end{aligned}
$$

- **Dirichlet** condition transform to **Robin** type condition !
- Forward-backward **might be** less important ?

Alternating Schwarz in time - Variants

Exchange p and y



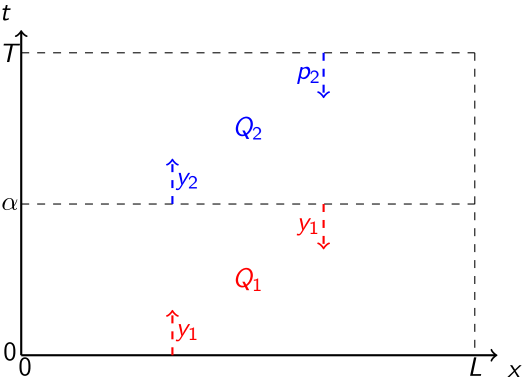
In $Q_1 = (0, L) \times (0, \alpha)$

$$\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\
y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\
y_1^\ell(x, 0) &= y_0(x) & y_1^\ell(x, \alpha) &= y_2^{\ell-1}(x, \alpha)
\end{aligned}$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\
y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\
y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\
p_2^\ell(x, \alpha) &= p_1^\ell(x, \alpha) & p_2^\ell(x, T) &= 0
\end{aligned}$$

Only with y



In $Q_1 = (0, L) \times (0, \alpha)$

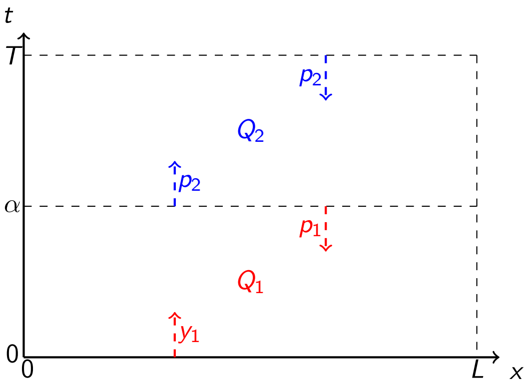
$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\
y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\
y_1^\ell(x, 0) &= y_0(x) & y_1^\ell(x, \alpha) &= y_2^{\ell-1}(x, \alpha)
\end{aligned}
$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$
\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\
y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\
y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\
y_2^\ell(x, \alpha) &= y_1^\ell(x, \alpha) & p_2^\ell(x, T) &= 0x
\end{aligned}
$$

Alternating Schwarz in time - Variants

Only with p



In $Q_1 = (0, L) \times (0, \alpha)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\
y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\
y_1^\ell(x, 0) &= y_0(x) & p_1^\ell(x, \alpha) &= p_2^{\ell-1}(x, \alpha)
\end{aligned}
$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$
\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\
y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\
y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\
p_2^\ell(x, \alpha) &= p_1^\ell(x, \alpha) & p_2^\ell(x, T) &= 0
\end{aligned}
$$

Dirichlet type transmission condition:

name	SD_1	SD_2	SD_3	SD_4
Q_1	p	y	y	p
Q_2	y	p	y	p

Alternating Schwarz in time - Convergence

Dirichlet type transmission condition:

name	SD_1	SD_2	SD_3	SD_4
Q_1	p	y	y	p
Q_2	y	p	y	p

Analysis: eigendecomposition in space

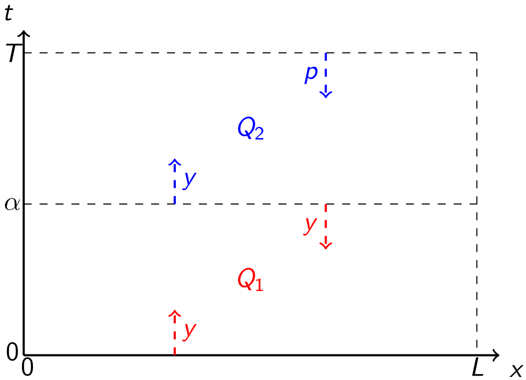
$$\rho_{SD_1} = \max_{\lambda_m \in \Lambda} \frac{1}{\nu(\sigma_m \coth(a_m) + \lambda_m)(\sigma_m \coth(b_m) + \lambda_m)}$$

$$\rho_{SD_2} = \max_{\lambda_m \in \Lambda} \nu(\sigma_m \coth(a_m) + \lambda_m)(\sigma_m \coth(b_m) + \lambda_m)$$

$$\rho_{SD_3} = 1$$

$$\rho_{SD_4} = 1$$

Only with y



In $Q_1 = (0, L) \times (0, \alpha)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\
y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\
y_1^\ell(x, 0) &= y_0(x) & y_1^\ell(x, \alpha) &= y_2^{\ell-1}(x, \alpha)
\end{aligned}
$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$
\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\
y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\
y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\
y_2^\ell(x, \alpha) &= y_1^\ell(x, \alpha) & p_2^\ell(x, T) &= 0
\end{aligned}
$$

Alternating Schwarz in time - Convergence

Dirichlet type transmission condition:

name	SD_1	SD_2	SD_3	SD_4
Q_1	p	y	y	p
Q_2	y	p	y	p

Analysis: eigendecomposition in space

$$\rho_{\mathrm{SD}_1} = \max_{\lambda_m \in \Lambda} \frac{1}{\nu(\sigma_m \coth(a_m) + \lambda_m)(\sigma_m \coth(b_m) + \lambda_m)}$$

$$\rho_{\mathrm{SD}_2} = \max_{\lambda_m \in \Lambda} \nu(\sigma_m \coth(a_m) + \lambda_m)(\sigma_m \coth(b_m) + \lambda_m)$$

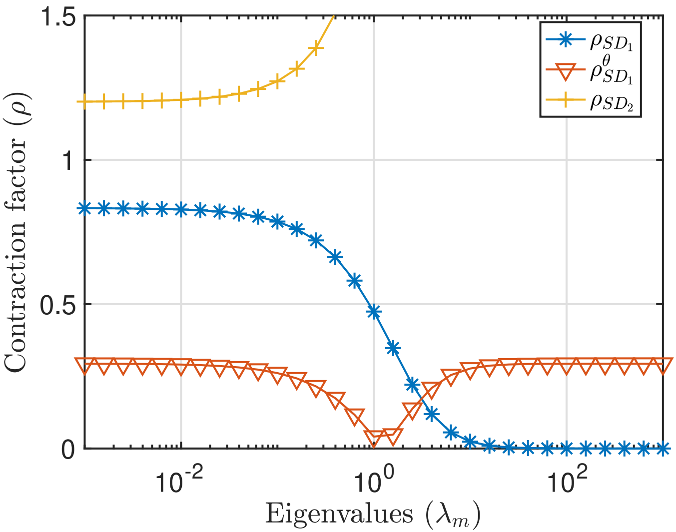
$$\rho_{\mathrm{SD}_3} = 1$$

$$\rho_{\mathrm{SD}_4} = 1$$

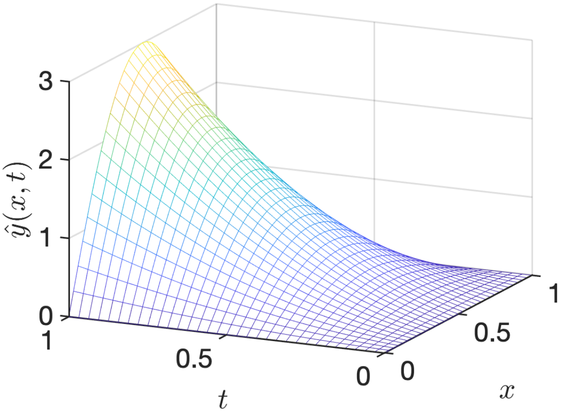
Relaxation: *e.g.*, $p_1^\ell(x,\alpha) = p_2^{\ell-1}(x,\alpha)$ to $p_1^\ell(x,\alpha) = f^\ell(x)$ with

$$f^{\ell+1}(x) = (1-\theta)f^\ell(x) + \theta p_2^\ell(x,\alpha) \qquad \theta \in (0,1)$$

Alternating Schwarz in time - Contraction

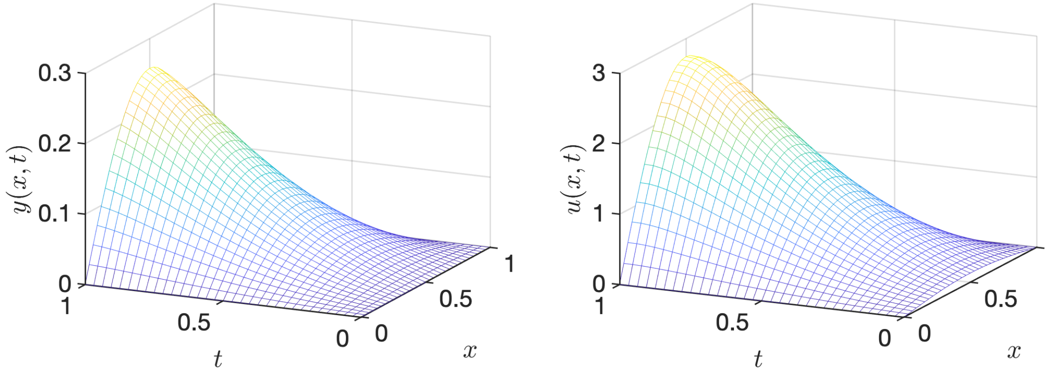


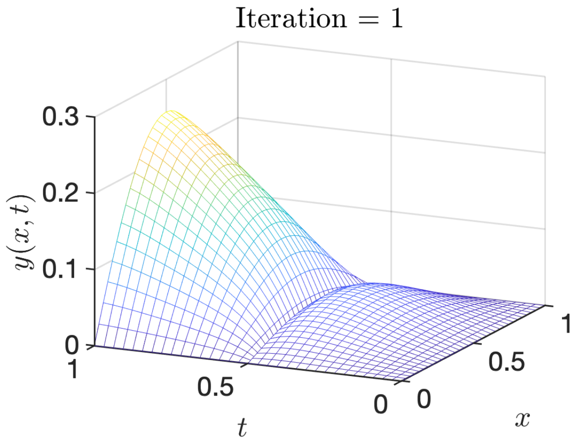
Target function: $\hat{y}(x,t) = \sin(\pi x)(2t^2 + t)$

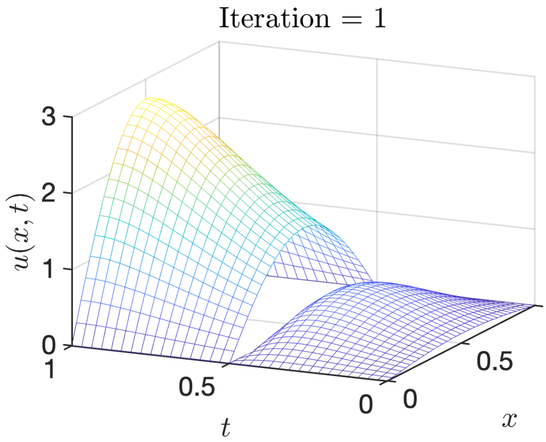


Numerical experiments

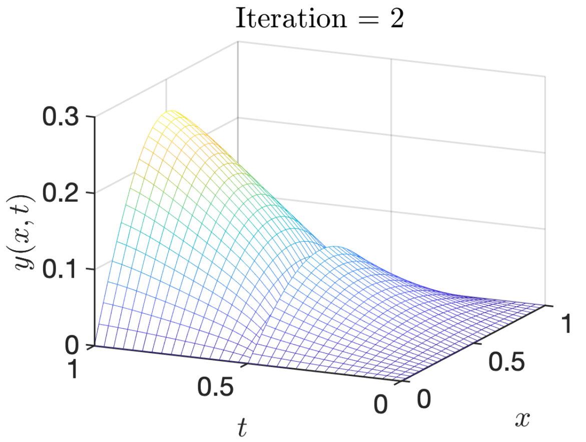
Numerical solutions: Crank-Nicolson with mesh size $h_t = h_x = \frac{1}{32}$ and penalization parameter: $\nu = 0.1$

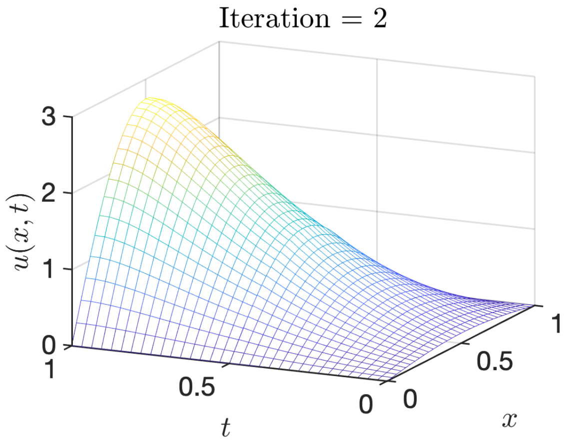


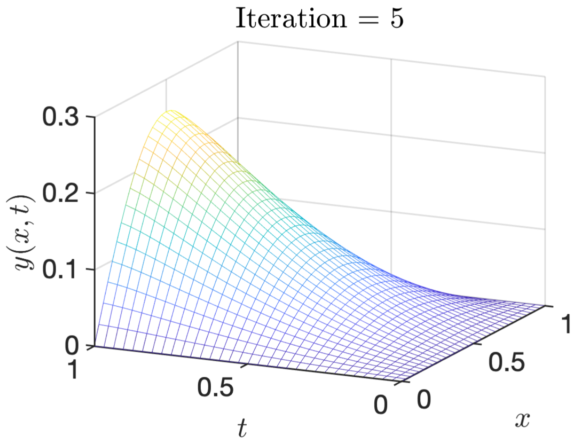

Iteration = 1
0.3
0.2
0.1
0
$y(x,t)$
1
0.5
0
t
0
0.5
1
x

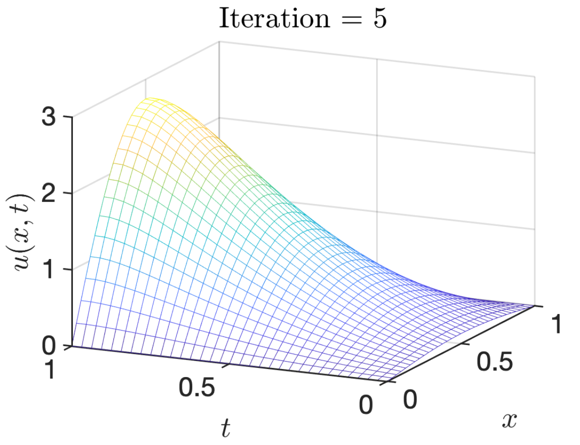

Iteration = 1
3
2
1
0
$u(x,t)$
1
0.5
0
t
0
0.5
1
x

Numerical experiments

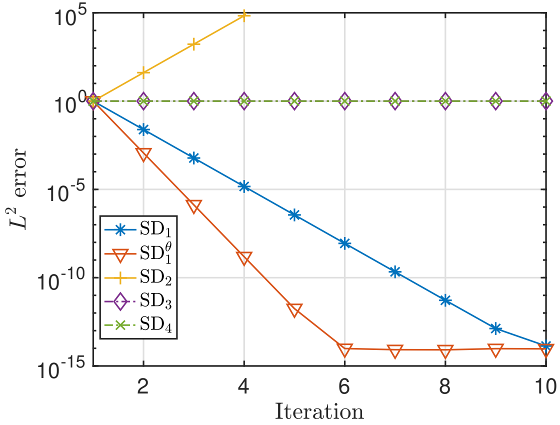


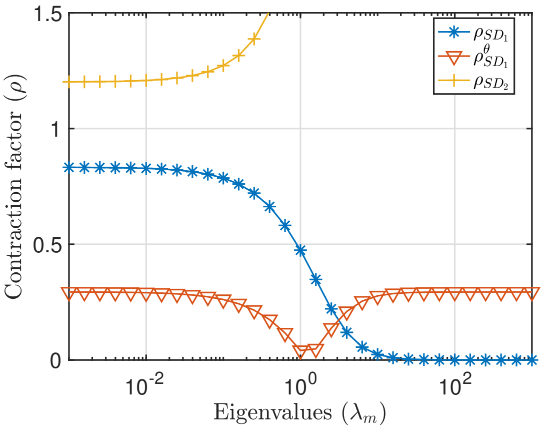



Iteration = 5
$y(x,t)$
0.3
0.2
0.1
0
1
0.5
0
t
0
0.5
1
x


Iteration = 5
$u(x,t)$
3
2
1
0
1
0.5
0
t
0
0.5
1
x

Numerical experiments - Convergence behavior





What we learned

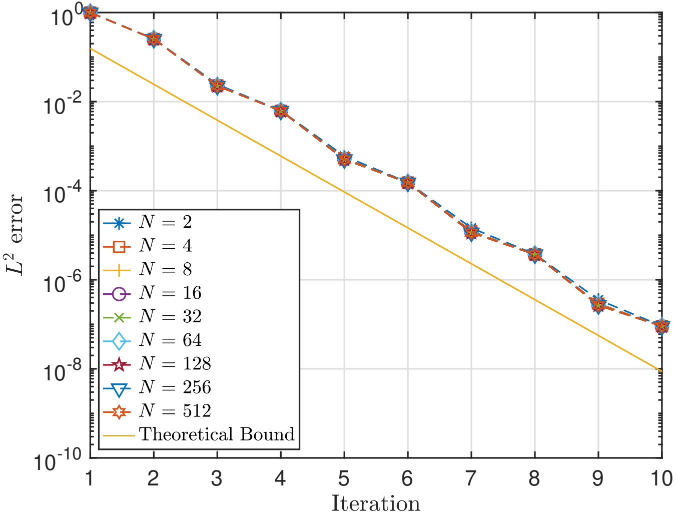
Three main messages:

- Time decomposition is **different** from space decomposition
- Forward-backward structure is **important** for alternating Schwarz to converge without overlap
- Relaxation **can improve** the convergence, but is **not necessary** for the method to converge

Many other variants: 3 Dirichlet–Neumann, 9 Neumann–Neumann, 36 Optimized Schwarz, ...

Gander, L., SINUM, 62(4):2048–2070, 2024 Gander, L., SINUM, 62(6):2588–2610, 2024
Gander, L., Domain Decomposition Methods in Science and Engineering XXVIII, 155:367–374, 2026

Excellent weak scalability



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Excellent weak scalability: Detailed analysis in L., Vanzan, arXiv:2603.07768

Thank you for your attention !