Time domain decomposition for PDE-constrained optimization

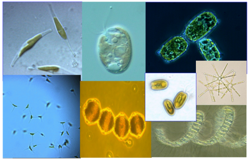
Liu-Di LU

Matematikcentrums
Lunds Universitet

International Workshop on PDE-constrained Optimization: Theory, Numerics and Applications

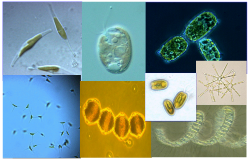
Tianyuan Mathematics Research Center
June 18, 2026

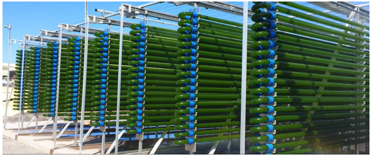
Application: biomass production



Microalgae: photosynthetic micro-organisms 2-50 μm in acquatic environment

Application: biomass production



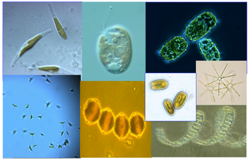


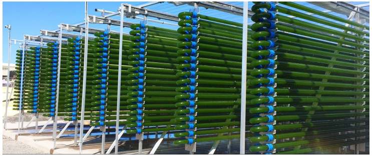
Microalgae: photosynthetic micro-organisms 2-50 μm in acquatic environment

Photobioreactors:

- technology: chemostats, raceway, tubular, etc
- position: indoor/outdoor
- scale: lab/industrial
- purpose: wastewater treatment, biofuel production, etc

Application: biomass production



Microalgae: photosynthetic micro-organisms 2-50 μm in acquatic environment

Photobioreactors:

- technology: chemostats, raceway, tubular, etc
- position: indoor/outdoor
- scale: lab/industrial
- purpose: wastewater treatment, biofuel production, etc

Challenges:

- both physical and biological behavior
- coupled models
- different timescales
- macro/micro levels
- uncertainties: environment, measurement, etc

Example: chemostats

Maximize biomass productivity $P(X)$

Governing equation: reaction-diffusion

$$\partial_t X - \kappa \Delta X + R(X) = f$$

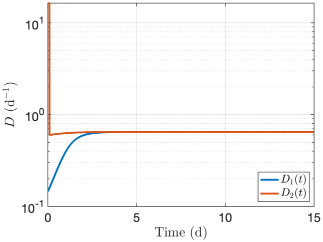
Example: chemostats

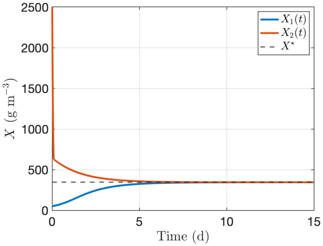
Maximize biomass productivity $P(X)$

Governing equation: reaction-diffusion

$$\partial_t X - \kappa \Delta X + R(X) = f$$

e.g., control with the dilution rate $R(X) = D(X)$





Example: raceway ponds

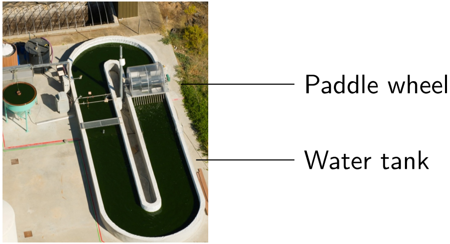
Paddle wheel

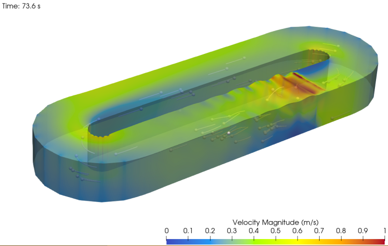
Water tank

Maximize average growth rate $\bar{\mu}(X)$

Governing physics: hydrodynamics

Navier-Stokes, Shallow water, etc





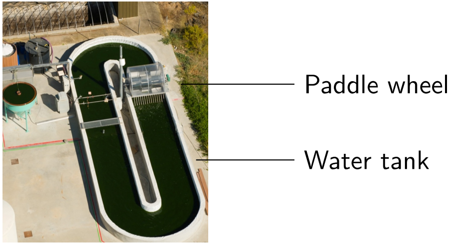
Example: raceway ponds

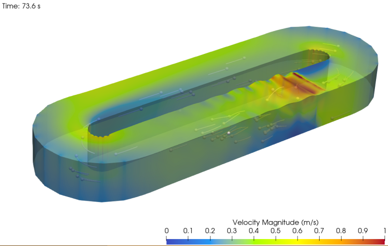
Maximize average growth rate $\bar{\mu}(X)$

Governing physics: hydrodynamics
  Navier-Stokes, Shallow water, etc

Microalgae movement: advection-diffusion

$$\partial_t X + \boldsymbol{u} \cdot \nabla X - \kappa \Delta X = f$$





Example: raceway ponds

Maximize average growth rate $\bar{\mu}(X)$

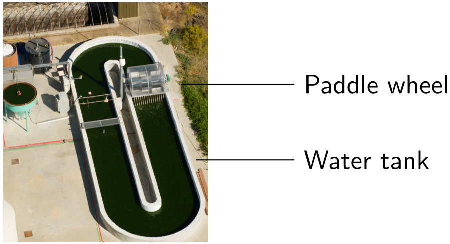
Governing physics: hydrodynamics

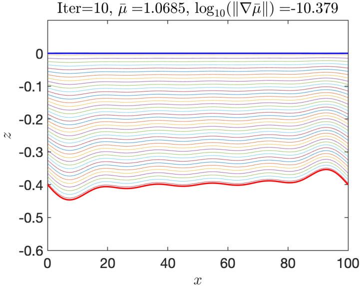
Navier-Stokes, Shallow water, etc

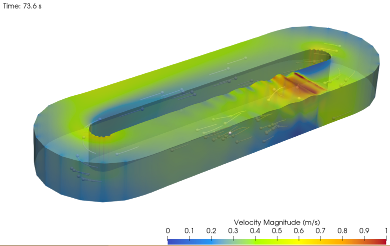
Microalgae movement: advection-diffusion

$$\partial_t X + \boldsymbol{u} \cdot \nabla X - \kappa \Delta X = f$$

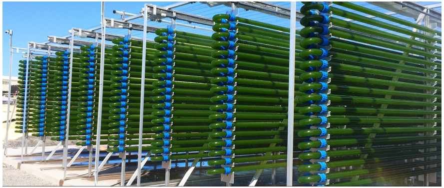
e.g., design best shape of the topography

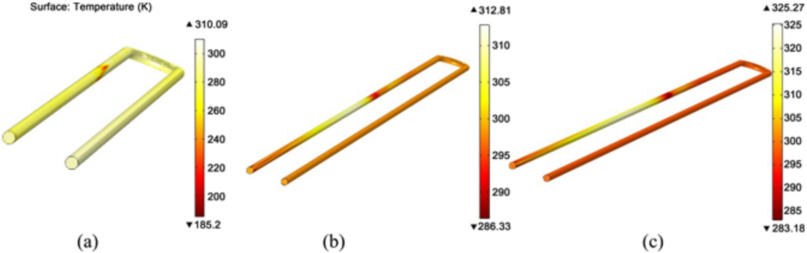






Example: tubular photobioreactor





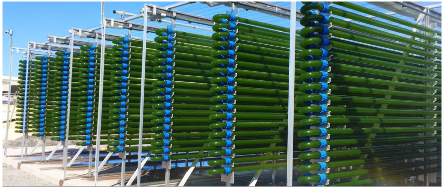
Example: tubular photobioreactor

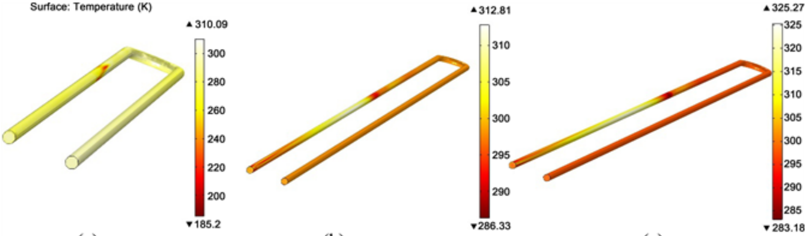
Minimize temperature distribution w.r.t. target $\hat{y}$

$$J(y,u) = \frac{1}{2}\|y-\hat{y}\|^2_{L^2(Q)} + \frac{\nu}{2}\|u\|^2_{L^2(Q)}$$

Governing equation: heat equation

$$\partial_t y - \kappa \Delta y = u$$





(a) (b) (c)

Model problem

Temperature control w.r.t. a target:

minimize the functional $J(y,u) := \frac{1}{2}\|y-\hat{y}\|^2_{L^2(Q)} + \frac{\nu}{2}\|u\|^2_{L^2(Q)}$, $\hat{y} \in L^2(Q)$, $\nu > 0$

constrained by $\partial_t y - \Delta y = u$ in $Q := \Omega \times (0,T)$

completed with initial condition y_0 and boundary condition $y = 0$ on $\Sigma := \partial\Omega \times (0,T)$

Model problem

Temperature control w.r.t. a target:

minimize the functional $J(y,u) := \frac{1}{2}\|y-\hat{y}\|^2_{L^2(Q)} + \frac{\nu}{2}\|u\|^2_{L^2(Q)}$, $\hat{y} \in L^2(Q)$, $\nu > 0$

constrained by $\partial_t y - \Delta y = u$ in $Q := \Omega \times (0,T)$

completed with initial condition y_0 and boundary condition $y = 0$ on $\Sigma := \partial\Omega \times (0,T)$

Lagrange multiplier p:

$$\mathcal{L}(y,p,u) = J(y,u) + \langle p, \partial_t y - \Delta y - u\rangle$$

Derive first-order optimality system

$$\begin{aligned} \partial_t y - \Delta y &= u && \text{in } Q & \qquad \partial_t p + \Delta p &= y - \hat{y} && \text{in } Q \\ y &= 0 && \text{on } \Sigma & p &= 0 && \text{on } \Sigma, \\ y &= y_0 && \text{on } \Sigma_0 & p &= 0 && \text{on } \Sigma_T \end{aligned}$$

$$\nu u - p = 0 \quad \text{in } Q$$

Model problem

Reduced optimality system (forward-backward)

$$
\begin{aligned}
\partial_t y - \Delta y &= \nu^{-1} p && \text{in } Q \qquad & \partial_t p + \Delta p &= y - \hat{y} && \text{in } Q \\
y &= 0 && \text{on } \Sigma & p &= 0 && \text{on } \Sigma \\
y &= y_0 && \text{on } \Sigma_0 & p &= 0 && \text{on } \Sigma_T
\end{aligned}
$$

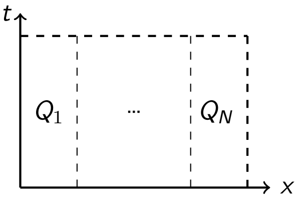
Model problem

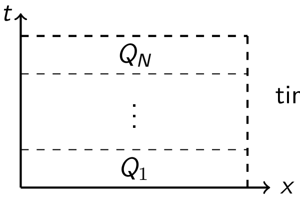
Reduced optimality system (forward-backward)

$$
\begin{aligned}
\partial_t y - \Delta y &= \nu^{-1} p && \text{in } Q \qquad & \partial_t p + \Delta p &= y - \hat{y} && \text{in } Q\\
y &= 0 && \text{on } \Sigma & p &= 0 && \text{on } \Sigma\\
y &= y_0 && \text{on } \Sigma_0 & p &= 0 && \text{on } \Sigma_T
\end{aligned}
$$

Apply domain decomposition methods

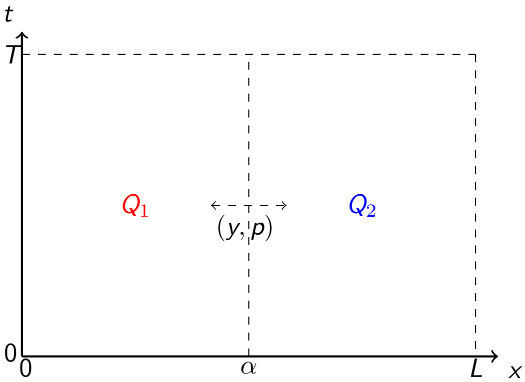
space decomposition





time decomposition

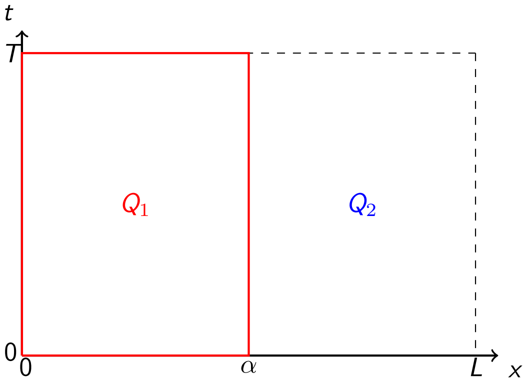
Space decomposition (Waveform method)



Subdomains: $Q_1 = (0, \alpha) \times (0, T)$ and $Q_2 = (\alpha, L) \times (0, T)$

$$
\begin{aligned}
\partial_t y - \partial_{xx} y &= \nu^{-1} p \qquad & \partial_t p + \partial_{xx} p &= y - \hat{y} \\
y(0, t) &= 0 & p(0, t) &= 0 \\
y(L, t) &= 0 & p(L, t) &= 0 \\
y(x, 0) &= y_0(x) & p(x, T) &= 0
\end{aligned}
$$

Alternating Schwarz waveform



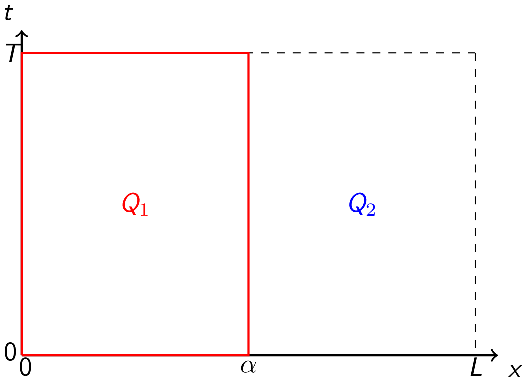
In $Q_1 = (0, \alpha) \times (0, T)$

$$\partial_t y_1^\ell - \partial_{xx} y_1^\ell = \nu^{-1} p_1^\ell \qquad \partial_t p_1^\ell + \partial_{xx} p_1^\ell = y_1^\ell - \hat{y}_1$$

$$y_1^\ell(0, t) = 0 \qquad p_1^\ell(0, t) = 0$$

$$y_1^\ell(x, 0) = y_{1,0}(x) \qquad p_1^\ell(x, T) = 0$$

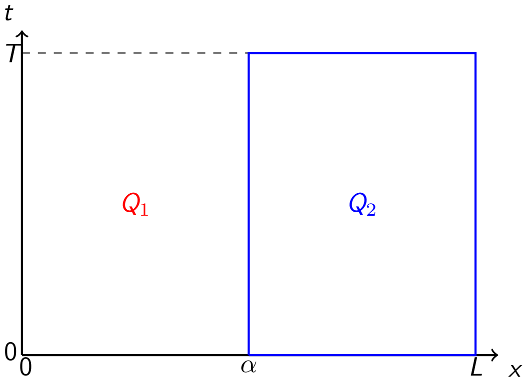
Alternating Schwarz waveform



In $Q_1 = (0,\alpha) \times (0, T)$

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0,t) &= 0 & p_1^\ell(0,t) &= 0 \\ y_1^\ell(\alpha,t) &= y_2^{\ell-1}(\alpha,t) & p_1^\ell(\alpha,t) &= p_2^{\ell-1}(\alpha,t) \\ y_1^\ell(x,0) &= y_{1,0}(x) & p_1^\ell(x,T) &= 0 \end{aligned}$$

Alternating Schwarz waveform



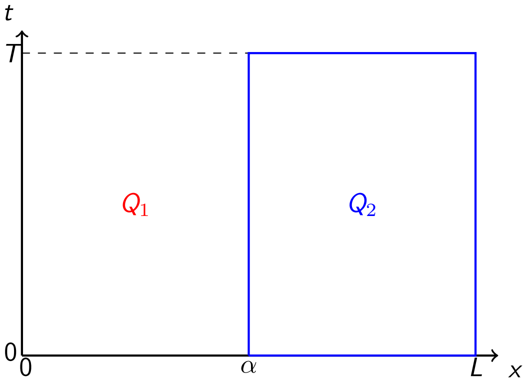
In $Q_1 = (0, \alpha) \times (0, T)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0,t) &= 0 & p_1^\ell(0,t) &= 0 \\
y_1^\ell(\alpha,t) &= y_2^{\ell-1}(\alpha,t) & p_1^\ell(\alpha,t) &= p_2^{\ell-1}(\alpha,t) \\
y_1^\ell(x,0) &= y_{1,0}(x) & p_1^\ell(x,T) &= 0
\end{aligned}
$$

In $Q_2 = (\alpha, L) \times (0, T)$

$$
\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\
y_2^\ell(L,t) &= 0 & p_2^\ell(L,t) &= 0 \\
y_2^\ell(x,0) &= y_{2,0}(x) & p_2^\ell(x,T) &= 0
\end{aligned}
$$

Alternating Schwarz waveform



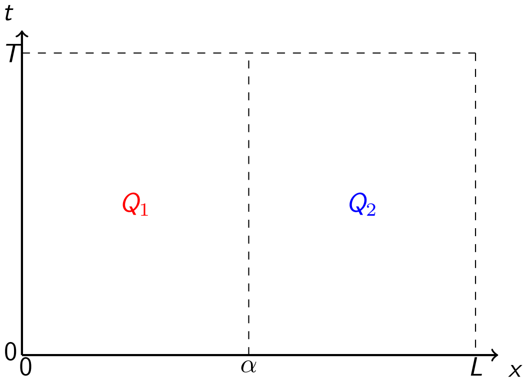
In $Q_1 = (0,\alpha)\times(0,T)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \qquad \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1\\
y_1^\ell(0,t) &= 0 & p_1^\ell(0,t) &= 0\\
y_1^\ell(\alpha,t) &= y_2^{\ell-1}(\alpha,t) & p_1^\ell(\alpha,t) &= p_2^{\ell-1}(\alpha,t)\\
y_1^\ell(x,0) &= y_{1,0}(x) & p_1^\ell(x,T) &= 0
\end{aligned}
$$

In $Q_2 = (\alpha,L)\times(0,T)$

$$
\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \qquad \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2\\
y_2^\ell(\alpha,t) &= y_1^\ell(\alpha,t) & p_2^\ell(\alpha,t) &= p_1^\ell(\alpha,t)\\
y_2^\ell(L,t) &= 0 & p_2^\ell(L,t) &= 0\\
y_2^\ell(x,0) &= y_{2,0}(x) & p_2^\ell(x,T) &= 0
\end{aligned}
$$

Alternating Schwarz waveform



Does not converge without overlap !

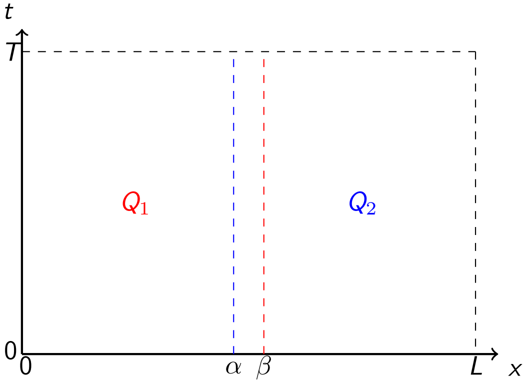
In $Q_1 = (0,\alpha) \times (0,T)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0,t) &= 0 & p_1^\ell(0,t) &= 0 \\
y_1^\ell(\alpha,t) &= y_2^{\ell-1}(\alpha,t) & p_1^\ell(\alpha,t) &= p_2^{\ell-1}(\alpha,t) \\
y_1^\ell(x,0) &= y_{1,0}(x) & p_1^\ell(x,T) &= 0
\end{aligned}
$$

In $Q_2 = (\alpha,L) \times (0,T)$

$$
\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\
y_2^\ell(\alpha,t) &= y_1^\ell(\alpha,t) & p_2^\ell(\alpha,t) &= p_1^\ell(\alpha,t) \\
y_2^\ell(L,t) &= 0 & p_2^\ell(L,t) &= 0 \\
y_2^\ell(x,0) &= y_{2,0}(x) & p_2^\ell(x,T) &= 0
\end{aligned}
$$

Alternating Schwarz waveform



Now converges with overlap !

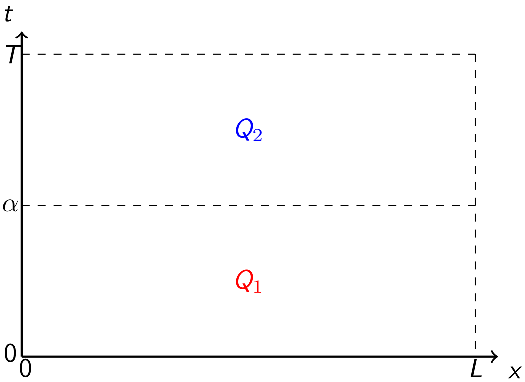
In $Q_1 = (0, \beta) \times (0, T)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\
y_1^\ell(\beta, t) &= y_2^{\ell-1}(\beta, t) & p_1^\ell(\beta, t) &= p_2^{\ell-1}(\beta, t) \\
y_1^\ell(x, 0) &= y_{1,0}(x) & p_1^\ell(x, T) &= 0
\end{aligned}
$$

In $Q_2 = (\alpha, L) \times (0, T)$

$$
\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\
y_2^\ell(\alpha, t) &= y_1^\ell(\alpha, t) & p_2^\ell(\alpha, t) &= p_1^\ell(\alpha, t) \\
y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\
y_2^\ell(x, 0) &= y_{2,0}(x) & p_2^\ell(x, T) &= 0
\end{aligned}
$$

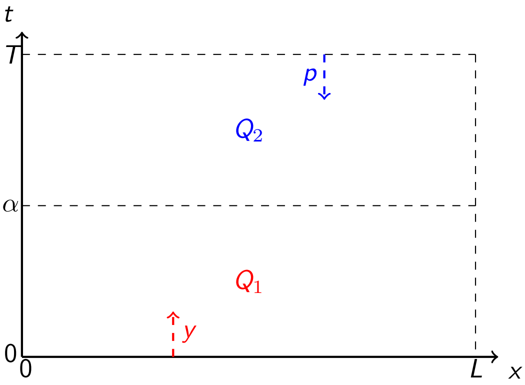
Time decomposition



Subdomains: $Q_1 = (0, L) \times (0, \alpha)$ and $Q_2 = (0, L) \times (\alpha, T)$

$$
\begin{aligned}
\partial_t y - \partial_{xx} y &= \nu^{-1} p & \qquad \partial_t p + \partial_{xx} p &= y - \hat{y} \\
y(0, t) &= 0 & p(0, t) &= 0 \\
y(L, t) &= 0 & p(L, t) &= 0 \\
y(x, 0) &= y_0(x) & p(x, T) &= 0
\end{aligned}
$$

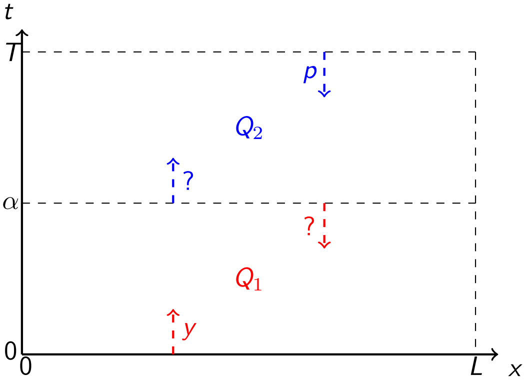
Time decomposition



Subdomains: $Q_1 = (0, L) \times (0, \alpha)$ and $Q_2 = (0, L) \times (\alpha, T)$

$$
\begin{aligned}
\partial_t y - \partial_{xx} y &= \nu^{-1} p & \qquad \partial_t p + \partial_{xx} p &= y - \hat{y} \\
y(0, t) &= 0 & p(0, t) &= 0 \\
y(L, t) &= 0 & p(L, t) &= 0 \\
y(x, 0) &= y_0(x) & p(x, T) &= 0
\end{aligned}
$$

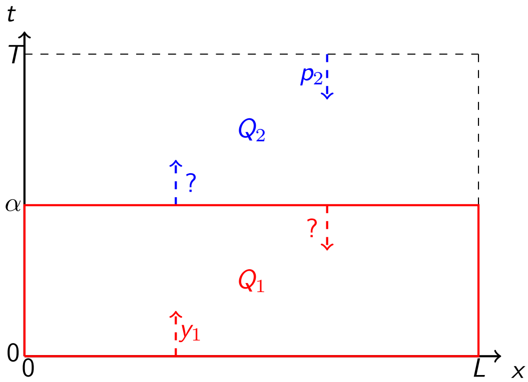
Time decomposition



Subdomains: $Q_1 = (0, L) \times (0, \alpha)$ and $Q_2 = (0, L) \times (\alpha, T)$

$$
\begin{aligned}
\partial_t y - \partial_{xx} y &= \nu^{-1} p & \qquad \partial_t p + \partial_{xx} p &= y - \hat{y} \\
y(0, t) &= 0 & p(0, t) &= 0 \\
y(L, t) &= 0 & p(L, t) &= 0 \\
y(x, 0) &= y_0(x) & p(x, T) &= 0
\end{aligned}
$$

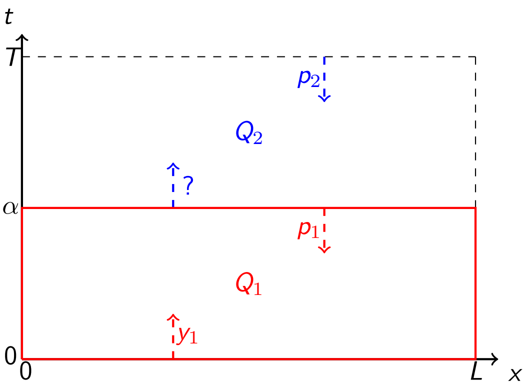
Alternating Schwarz in time



In $Q_1 = (0, L) \times (0, \alpha)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\
y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\
y_1^\ell(x, 0) &= y_0(x) & &
\end{aligned}
$$

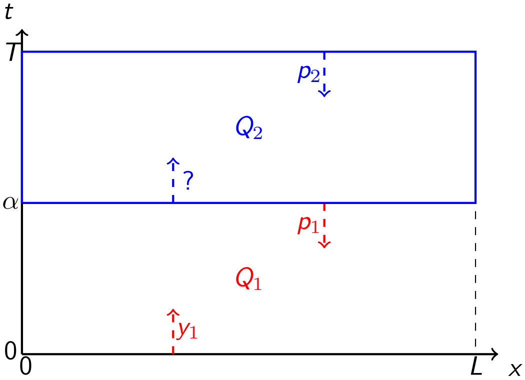
Alternating Schwarz in time



In $Q_1 = (0, L) \times (0, \alpha)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\
y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\
y_1^\ell(x, 0) &= y_0(x) & p_1^\ell(x, \alpha) &= p_2^{\ell-1}(x, \alpha)
\end{aligned}
$$

Alternating Schwarz in time



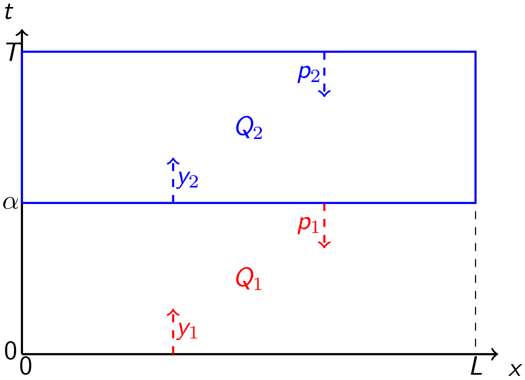
In $Q_1 = (0, L) \times (0, \alpha)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\
y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\
y_1^\ell(x, 0) &= y_0(x) & p_1^\ell(x, \alpha) &= p_2^{\ell-1}(x, \alpha)
\end{aligned}
$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$
\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\
y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\
y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\
& & p_2^\ell(x, T) &= 0
\end{aligned}
$$

Alternating Schwarz in time

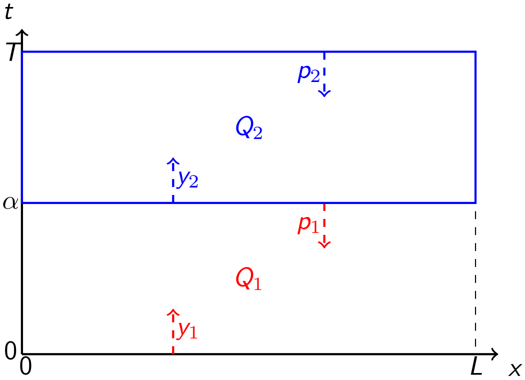


In $Q_1 = (0, L) \times (0, \alpha)$

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\ y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\ y_1^\ell(x, 0) &= y_0(x) & p_1^\ell(x, \alpha) &= p_2^{\ell-1}(x, \alpha) \end{aligned}$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\ y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\ y_2^\ell(x, \alpha) &= y_1^\ell(x, \alpha) & p_2^\ell(x, T) &= 0 \end{aligned}$$



How about the convergence ?

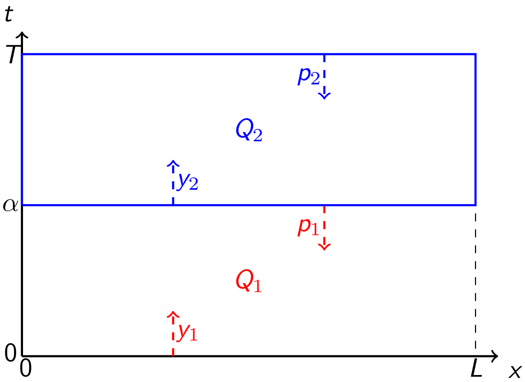
In $Q_1 = (0, L) \times (0, \alpha)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0,t) &= 0 & p_1^\ell(0,t) &= 0 \\
y_1^\ell(L,t) &= 0 & p_1^\ell(L,t) &= 0 \\
y_1^\ell(x,0) &= y_0(x) & p_1^\ell(x,\alpha) &= p_2^{\ell-1}(x,\alpha)
\end{aligned}
$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$
\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\
y_2^\ell(0,t) &= 0 & p_2^\ell(0,t) &= 0 \\
y_2^\ell(L,t) &= 0 & p_2^\ell(L,t) &= 0 \\
y_2^\ell(x,\alpha) &= y_1^\ell(x,\alpha) & p_2^\ell(x,T) &= 0
\end{aligned}
$$

Alternating Schwarz in time



How about the convergence ?
It **does converge** without overlap !

In $Q_1 = (0, L) \times (0, \alpha)$

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\ y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\ y_1^\ell(x, 0) &= y_0(x) & p_1^\ell(x, \alpha) &= p_2^{\ell-1}(x, \alpha) \end{aligned}$$

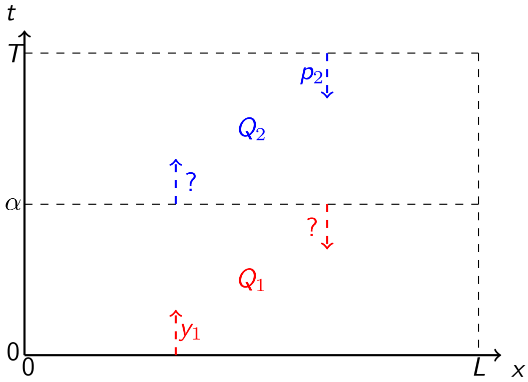
In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\ y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\ y_2^\ell(x, \alpha) &= y_1^\ell(x, \alpha) & p_2^\ell(x, T) &= 0 \end{aligned}$$

Alternating Schwarz in time - Intuition

Relations

$$p_j^\ell = \nu(\partial_t y_j^\ell - \partial_{xx} y_j^\ell)$$
$$y_j^\ell = \partial_t p_j^\ell + \partial_{xx} p_j^\ell + \hat{y}_j$$



In $Q_1 = (0, L) \times (0, \alpha)$

$$\partial_t y_1^\ell - \partial_{xx} y_1^\ell = \nu^{-1} p_1^\ell \quad \partial_t p_1^\ell + \partial_{xx} p_1^\ell = y_1^\ell - \hat{y}_1$$

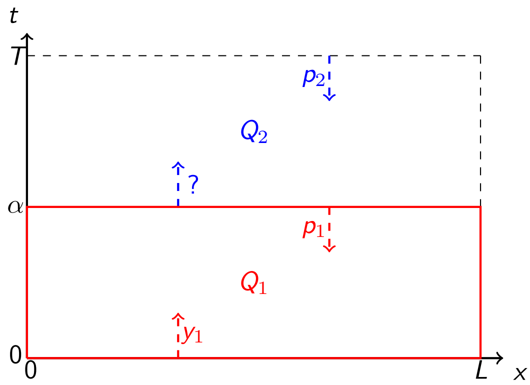
In $Q_2 = (0, L) \times (\alpha, T)$

$$\partial_t y_2^\ell - \partial_{xx} y_2^\ell = \nu^{-1} p_2^\ell \quad \partial_t p_2^\ell + \partial_{xx} p_2^\ell = y_2^\ell - \hat{y}_2$$

Alternating Schwarz in time - Intuition

Relations

$$p_j^\ell = \nu(\partial_t y_j^\ell - \partial_{xx} y_j^\ell)$$



In $Q_1 = (0, L) \times (0, \alpha)$

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0,t) &= 0 & p_1^\ell(0,t) &= 0 \\ y_1^\ell(L,t) &= 0 & p_1^\ell(L,t) &= 0 \\ y_1^\ell(x,0) &= y_0(x) & p_1^\ell(x,\alpha) &= p_2^{\ell-1}(x,\alpha) \end{aligned}$$

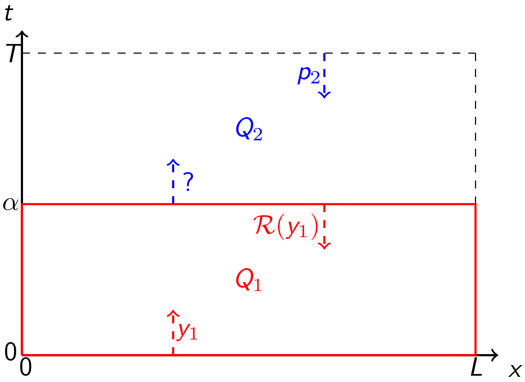
In $Q_2 = (0, L) \times (\alpha, T)$

$$\partial_t y_2^\ell - \partial_{xx} y_2^\ell = \nu^{-1} p_2^\ell \quad \partial_t p_2^\ell + \partial_{xx} p_2^\ell = y_2^\ell - \hat{y}_2$$

Alternating Schwarz in time - Intuition

Relations

$$p_j^\ell = \nu(\partial_t y_j^\ell - \partial_{xx} y_j^\ell)$$



In $Q_1 = (0, L) \times (0, \alpha)$

$$\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\
y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\
y_1^\ell(x, 0) &= y_0(x) & & \\
(\partial_t - \partial_{xx}) y_1^\ell(x, \alpha) &= (\partial_t - \partial_{xx}) y_2^{\ell-1}(x, \alpha) & &
\end{aligned}$$

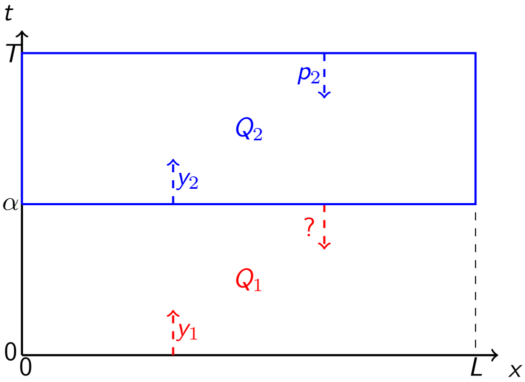
In $Q_2 = (0, L) \times (\alpha, T)$

$$\partial_t y_2^\ell - \partial_{xx} y_2^\ell = \nu^{-1} p_2^\ell \quad \partial_t p_2^\ell + \partial_{xx} p_2^\ell = y_2^\ell - \hat{y}_2$$

Alternating Schwarz in time - Intuition

Relations

$$y_j^\ell = \partial_t p_j^\ell + \partial_{xx} p_j^\ell + \hat{y}_j$$



In $Q_1 = (0, L) \times (0, \alpha)$

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\ y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\ y_1^\ell(x, 0) &= y_0(x) & & \end{aligned}$$

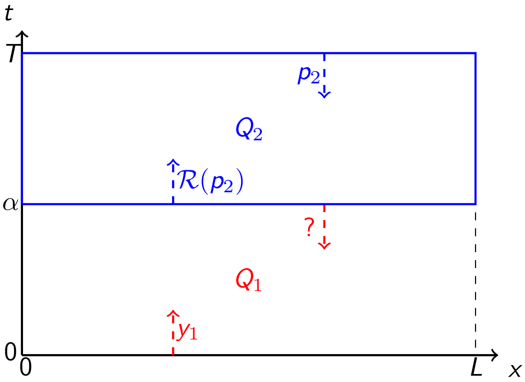
In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\ y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\ y_2^\ell(x, \alpha) &= y_1^\ell(x, \alpha) & p_2^\ell(x, T) &= 0 \end{aligned}$$

Alternating Schwarz in time - Intuition

Relations

$$y_j^\ell = \partial_t p_j^\ell + \partial_{xx} p_j^\ell + \hat{y}_j$$



In $Q_1 = (0, L) \times (0, \alpha)$

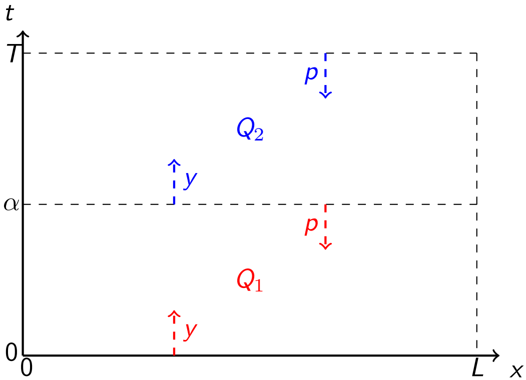
$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0,t) &= 0 & p_1^\ell(0,t) &= 0 \\ y_1^\ell(L,t) &= 0 & p_1^\ell(L,t) &= 0 \\ y_1^\ell(x,0) &= y_0(x) \end{aligned}$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0,t) &= 0 & p_2^\ell(0,t) &= 0 \\ y_2^\ell(L,t) &= 0 & p_2^\ell(L,t) &= 0 \\ & & p_2^\ell(x,T) &= 0 \end{aligned}$$

$$(\partial_t + \partial_{xx}) p_2^\ell(x,\alpha) = (\partial_t + \partial_{xx}) p_1^\ell(x,\alpha)$$

Alternating Schwarz in time - Observation



- **Dirichlet** condition transform to **Robin** type condition !

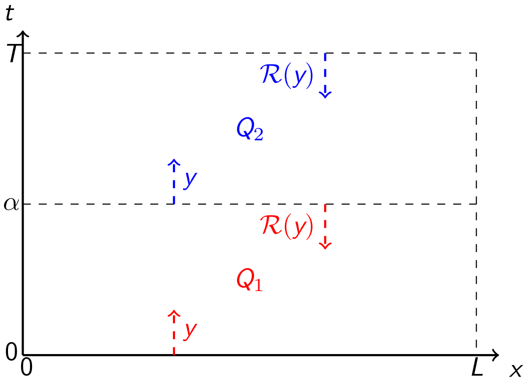
In $Q_1 = (0, L) \times (0, \alpha)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\
y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\
y_1^\ell(x, 0) &= y_0(x) & p_1^\ell(x, \alpha) &= p_2^{\ell-1}(x, \alpha)
\end{aligned}
$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$
\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\
y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\
y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\
y_2^\ell(x, \alpha) &= y_1^\ell(x, \alpha) & p_2^\ell(x, T) &= 0
\end{aligned}
$$

Alternating Schwarz in time - Observation



In $Q_1 = (0, L) \times (0, \alpha)$

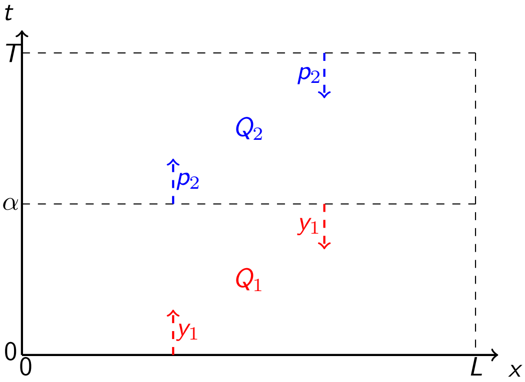
$$-\partial_{tt} y_1^\ell + \Delta^2 y_1^\ell + \nu^{-1} y_1^\ell = \nu^{-1}\hat{y}_1$$
$$y_1^\ell(x, 0) = y_0(x)$$
$$(\partial_t - \partial_{xx}) y_1^\ell(x, \alpha) = (\partial_t - \partial_{xx}) y_2^{\ell-1}(x, \alpha)$$
$$+ \text{ B.C.}$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$-\partial_{tt} y_2^\ell + \Delta^2 y_2^\ell + \nu^{-1} y_2^\ell = \nu^{-1}\hat{y}_2$$
$$y_2^\ell(x, \alpha) = y_1^\ell(x, \alpha)$$
$$(\partial_t - \partial_{xx}) y_2^{\ell-1}(x, T) = 0$$
$$+ \text{ B.C.}$$

- **Dirichlet** condition transform to **Robin** type condition !
- Forward-backward **might be** less important ?

Exchange p and y



In $Q_1 = (0, L) \times (0, \alpha)$

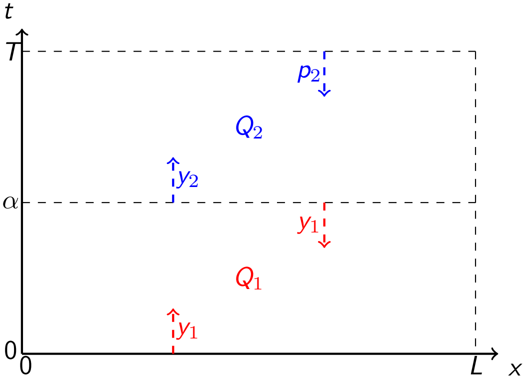
$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\ y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\ y_1^\ell(x, 0) &= y_0(x) & y_1^\ell(x, \alpha) &= y_2^{\ell-1}(x, \alpha) \end{aligned}$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\ y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\ p_2^\ell(x, \alpha) &= p_1^\ell(x, \alpha) & p_2^\ell(x, T) &= 0 \end{aligned}$$

Alternating Schwarz in time - Variants

Only with y



In $Q_1 = (0, L) \times (0, \alpha)$

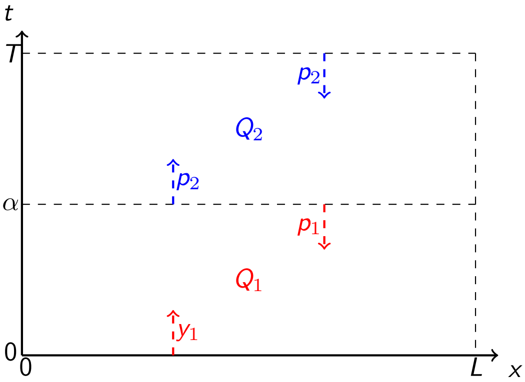
$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\
y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\
y_1^\ell(x, 0) &= y_0(x) & y_1^\ell(x, \alpha) &= y_2^{\ell-1}(x, \alpha)
\end{aligned}
$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$
\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\
y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\
y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\
y_2^\ell(x, \alpha) &= y_1^\ell(x, \alpha) & p_2^\ell(x, T) &= 0x
\end{aligned}
$$

Alternating Schwarz in time - Variants

Only with p



In $Q_1 = (0, L) \times (0, \alpha)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\
y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\
y_1^\ell(x, 0) &= y_0(x) & p_1^\ell(x, \alpha) &= p_2^{\ell-1}(x, \alpha)
\end{aligned}
$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$
\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\
y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\
y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\
p_2^\ell(x, \alpha) &= p_1^\ell(x, \alpha) & p_2^\ell(x, T) &= 0
\end{aligned}
$$

Alternating Schwarz in time - Convergence

Dirichlet type transmission condition:

name	SD_1	SD_2	SD_3	SD_4
Q_1	p	y	y	p
Q_2	y	p	y	p

Dirichlet type transmission condition:

name	SD_1	SD_2	SD_3	SD_4
Q_1	p	y	y	p
Q_2	y	p	y	p

Analysis: eigendecomposition in space

$$\rho_{\mathsf{SD}_1} = \max_{\lambda_m \in \Lambda} \frac{1}{\nu(\sigma_m \coth(a_m) + \lambda_m)(\sigma_m \coth(b_m) + \lambda_m)}$$

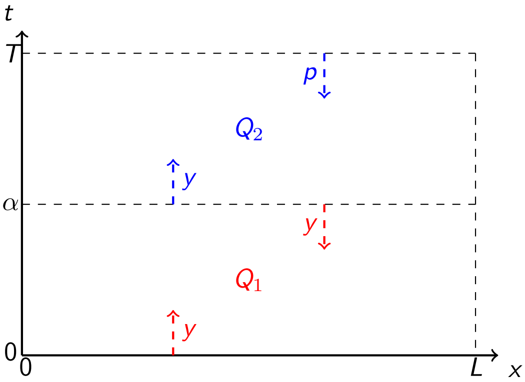
$$\rho_{\mathsf{SD}_2} = \max_{\lambda_m \in \Lambda} \nu(\sigma_m \coth(a_m) + \lambda_m)(\sigma_m \coth(b_m) + \lambda_m)$$

$$\rho_{\mathsf{SD}_3} = 1$$

$$\rho_{\mathsf{SD}_4} = 1$$

Alternating Schwarz in time - Convergence

Only with y



In $Q_1 = (0, L) \times (0, \alpha)$

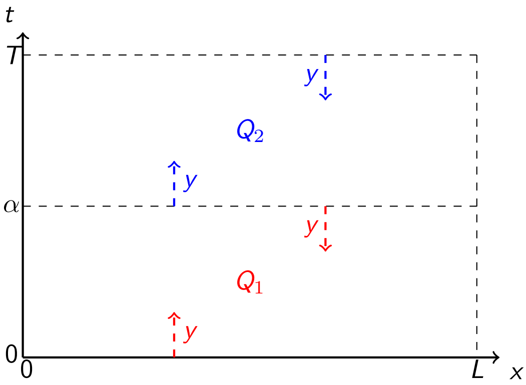
$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell & \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\ y_1^\ell(0, t) &= 0 & p_1^\ell(0, t) &= 0 \\ y_1^\ell(L, t) &= 0 & p_1^\ell(L, t) &= 0 \\ y_1^\ell(x, 0) &= y_0(x) & y_1^\ell(x, \alpha) &= y_2^{\ell-1}(x, \alpha) \end{aligned}$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} p_2^\ell & \partial_t p_2^\ell + \partial_{xx} p_2^\ell &= y_2^\ell - \hat{y}_2 \\ y_2^\ell(0, t) &= 0 & p_2^\ell(0, t) &= 0 \\ y_2^\ell(L, t) &= 0 & p_2^\ell(L, t) &= 0 \\ y_2^\ell(x, \alpha) &= y_1^\ell(x, \alpha) & p_2^\ell(x, T) &= 0x \end{aligned}$$

Alternating Schwarz in time - Convergence

Only with y



In $Q_1 = (0, L) \times (0, \alpha)$

$$
\begin{aligned}
-\partial_{tt} y_1^\ell + \Delta^2 y_1^\ell + \nu^{-1} y_1^\ell &= \nu^{-1} \hat{y}_1 \\
y_1^\ell(x, 0) &= y_0(x) \\
y_1^\ell(x, \alpha) &= y_2^{\ell-1}(x, \alpha) \\
&+ \text{ B.C.}
\end{aligned}
$$

In $Q_2 = (0, L) \times (\alpha, T)$

$$
\begin{aligned}
-\partial_{tt} y_2^\ell + \Delta^2 y_2^\ell + \nu^{-1} y_2^\ell &= \nu^{-1} \hat{y}_2 \\
y_2^\ell(x, \alpha) &= y_1^\ell(x, \alpha) \\
(\partial_t - \partial_{xx}) y_2^{\ell-1}(x, T) &= 0 \\
&+ \text{ B.C.}
\end{aligned}
$$

Alternating Schwarz in time - Convergence

Dirichlet type transmission condition:

name	SD_1	SD_2	SD_3	SD_4
Q_1	p	y	y	p
Q_2	y	p	y	p

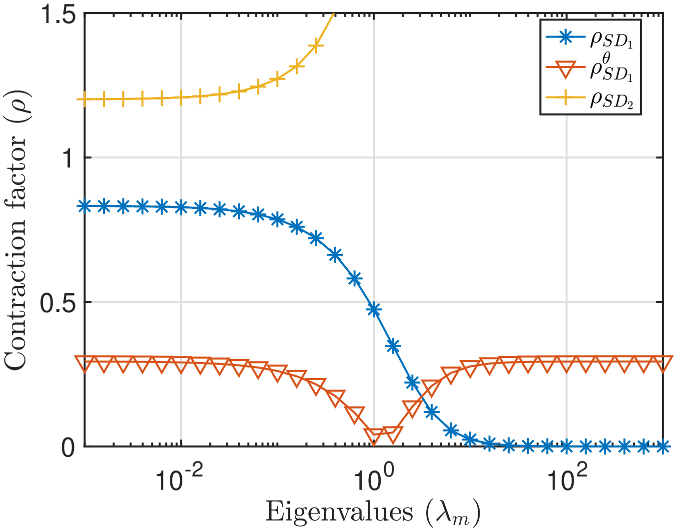
Analysis: eigendecomposition in space

$$
\begin{aligned}
\rho_{\text{SD}_1} &= \max_{\lambda_m \in \Lambda} \frac{1}{\nu(\sigma_m \coth(a_m) + \lambda_m)(\sigma_m \coth(b_m) + \lambda_m)} \\
\rho_{\text{SD}_2} &= \max_{\lambda_m \in \Lambda} \nu(\sigma_m \coth(a_m) + \lambda_m)(\sigma_m \coth(b_m) + \lambda_m) \\
\rho_{\text{SD}_3} &= 1 \\
\rho_{\text{SD}_4} &= 1
\end{aligned}
$$

Relaxation: *e.g.*, $p_1^\ell(x,\alpha) = p_2^{\ell-1}(x,\alpha)$ to $p_1^\ell(x,\alpha) = f^\ell(x)$ with

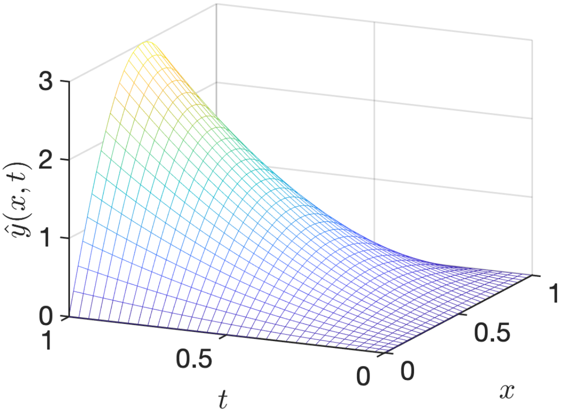
$$
f^{\ell+1}(x) = (1-\theta) f^\ell(x) + \theta p_2^\ell(x,\alpha) \qquad \theta \in (0,1)
$$

Alternating Schwarz in time - Contraction



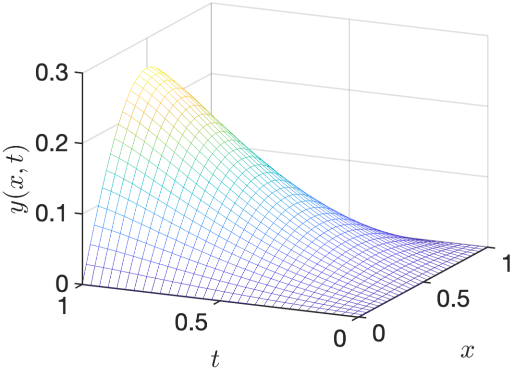
Numerical experiments

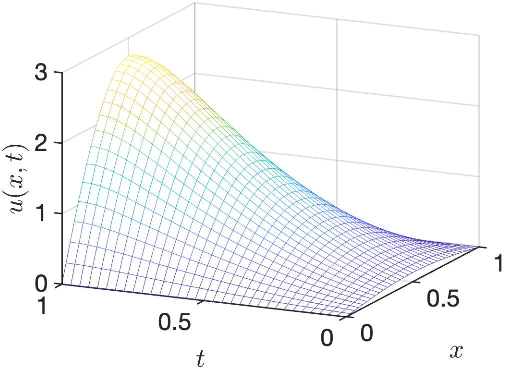
Target function: $\hat{y}(x,t) = \sin(\pi x)(2t^2 + t)$



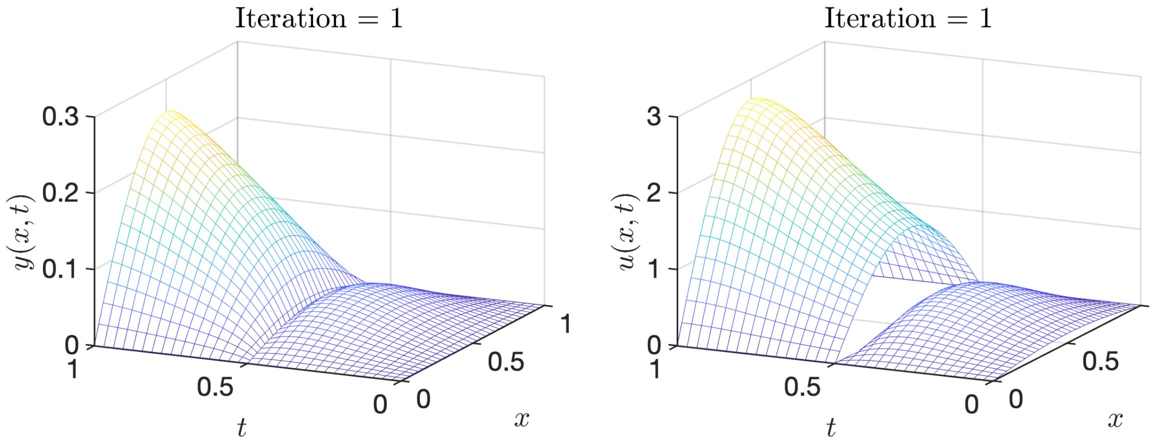
Numerical experiments

Numerical solutions: Crank-Nicolson with mesh size $h_t = h_x = \frac{1}{32}$ and penalization parameter: $\nu = 0.1$

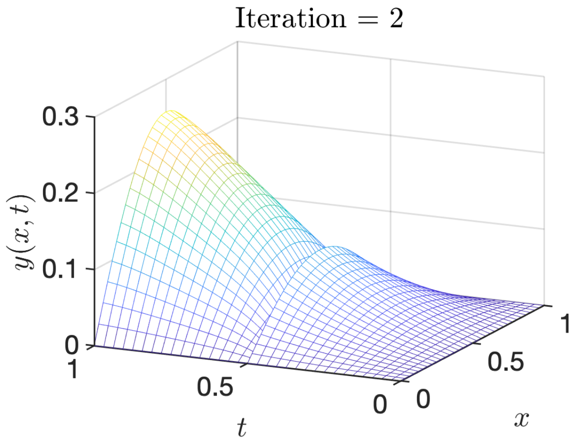


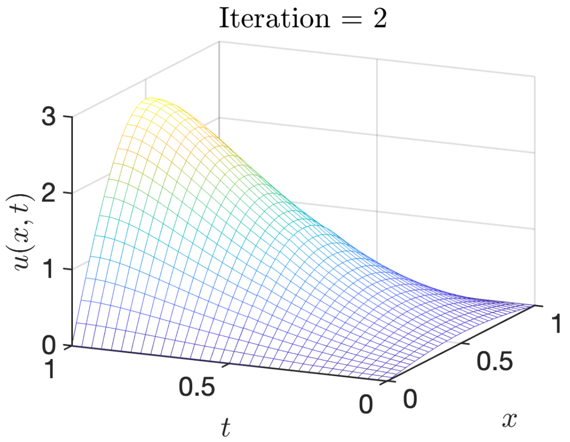


Numerical experiments

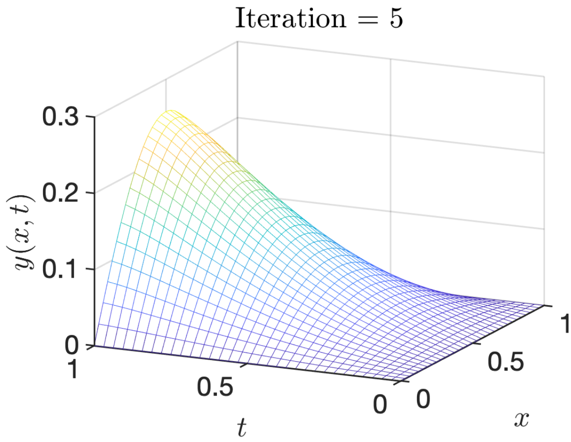


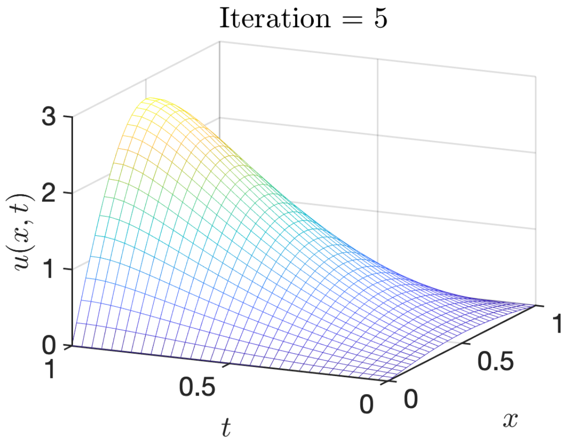
Numerical experiments



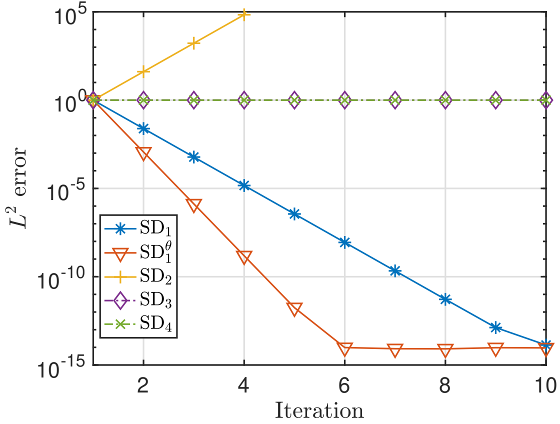


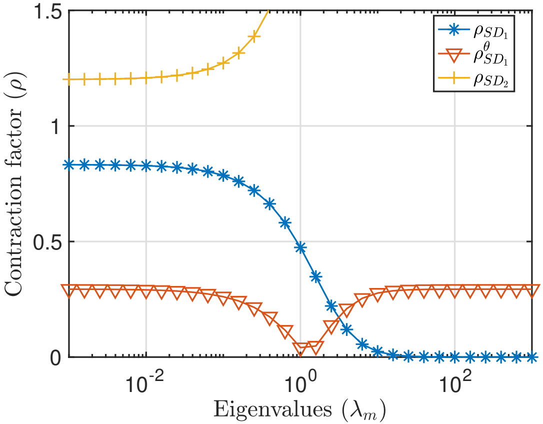
Numerical experiments





Numerical experiments - Convergence behavior





How about Neumann condition ?

Neumann type transmission condition:

name	SN_1	SN_2	SN_3	SN_4
Q_1	$\partial_t p$	$\partial_t y$	$\partial_t y$	$\partial_t p$
Q_2	$\partial_t y$	$\partial_t p$	$\partial_t y$	$\partial_t p$

How about Neumann condition ?

Neumann type transmission condition:

name	SN_1	SN_2	SN_3	SN_4
$\color{red}{Q_1}$	$\partial_t p$	$\partial_t y$	$\partial_t y$	$\partial_t p$
$\color{blue}{Q_2}$	$\partial_t y$	$\partial_t p$	$\partial_t y$	$\partial_t p$

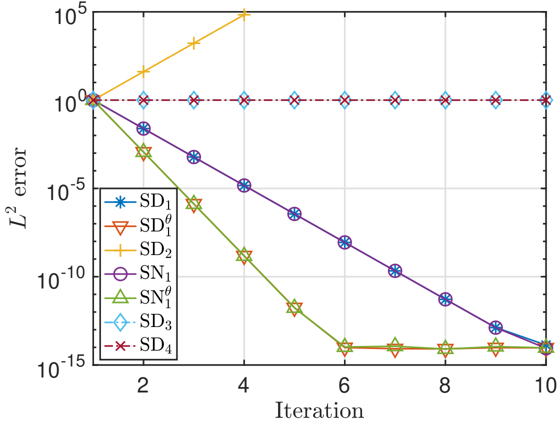
Analysis: eigendecomposition in space:

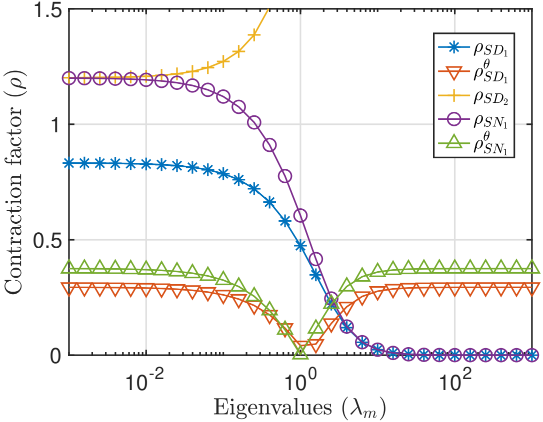
$$\begin{aligned}
\rho_{\mathsf{SN}_1} &= \max_{\lambda_m \in \Lambda} \frac{1}{\nu(\sigma_m \tanh(a_m) + \lambda_m)(\sigma_i \tanh(b_m) + \lambda_m)} \\
\rho_{\mathsf{SN}_2} &= \max_{\lambda_m \in \Lambda} \nu(\sigma_m \tanh(a_m) + \lambda_m)(\sigma_m \tanh(b_m) + \lambda_m) \\
\rho_{\mathsf{SN}_3} &= 1 \\
\rho_{\mathsf{SN}_4} &= 1
\end{aligned}$$

Relaxation: *e.g.*, $\partial_t p_1^\ell(x,\alpha) = \partial_t p_2^{\ell-1}(x,\alpha)$ to $\partial_t p_1^\ell(x,\alpha) = f^\ell(x)$ with

$$f^{\ell+1}(x) = (1-\theta) f^\ell(x) + \theta \partial_t p_2^\ell(x,\alpha) \qquad \theta \in (0,1)$$

How about Neumann condition ?





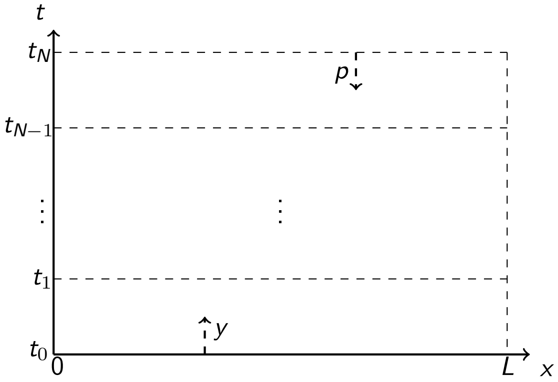
What we learned

Three main messages:

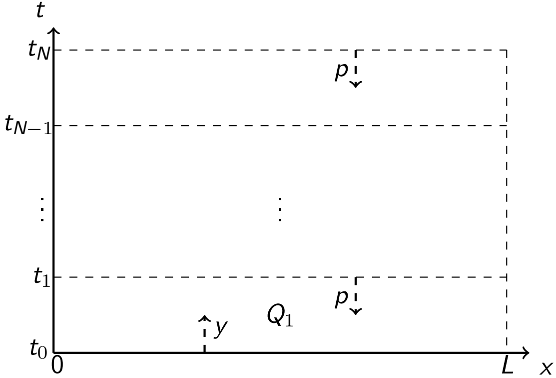
- Time decomposition is **different** from space decomposition
- Forward-backward structure is **important** for alternating Schwarz to converge without overlap
- Relaxation **can improve** the convergence, but is **not necessary** for the method to converge

Many other variants: 3 DN, 3 ND, 9 NN, 36 OS, ...

Many subdomains and scalability



Many subdomains and scalability



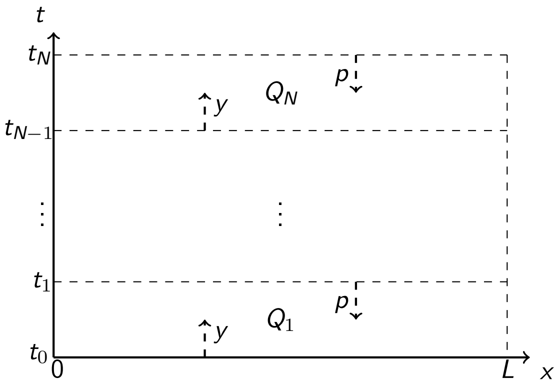
In $Q_1 = (0, L) \times (t_0, t_1)$

$$\partial_t y_1^\ell - \partial_{xx} y_1^\ell = \nu^{-1} p_1^\ell$$

$$\partial_t p_1^\ell + \partial_{xx} p_1^\ell = y_1^\ell - \hat{y}_1$$

$$y_1^\ell(x, t_0) = y_0(x), \quad p_1^\ell(x, t_1) = p_2^{\ell-1}(x, t_1)$$

Many subdomains and scalability



In $Q_1 = (0, L) \times (t_0, t_1)$

$$\partial_t y_1^\ell - \partial_{xx} y_1^\ell = \nu^{-1} p_1^\ell$$

$$\partial_t p_1^\ell + \partial_{xx} p_1^\ell = y_1^\ell - \hat{y}_1$$

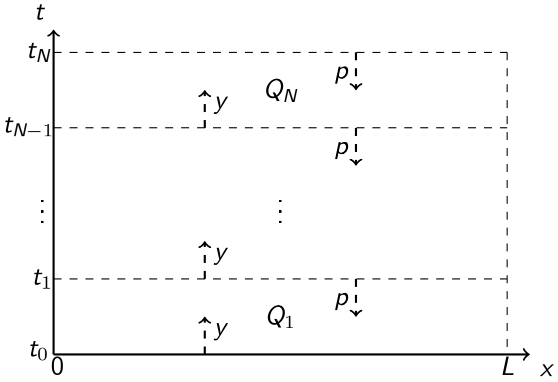
$$y_1^\ell(x, t_0) = y_0(x), \quad p_1^\ell(x, t_1) = p_2^{\ell-1}(x, t_1)$$

In $Q_N = (0, L) \times (t_{N-1}, T)$

$$\partial_t y_N^\ell - \partial_{xx} y_N^\ell = \nu^{-1} p_N^\ell$$

$$\partial_t p_N^\ell + \partial_{xx} p_N^\ell = y_N^\ell - \hat{y}_N$$

$$y_N^\ell(x, t_{N-1}) = y_N^{\ell-1}(x, t_{N-1}), \quad p_N^\ell(x, t_N) = 0$$



In $Q_1 = (0, L) \times (t_0, t_1)$

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell \\
\partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \\
y_1^\ell(x, t_0) = y_0(x), \quad p_1^\ell(x, t_1) &= p_2^{\ell-1}(x, t_1)
\end{aligned}
$$

In $Q_n = (0, L) \times (t_{n-1}, t_n)$

$$
\begin{aligned}
\partial_t y_n^\ell - \partial_{xx} y_n^\ell &= \nu^{-1} p_n^\ell \\
\partial_t p_n^\ell + \partial_{xx} p_n^\ell &= y_n^\ell - \hat{y}_n \\
y_n^\ell(x, t_{n-1}) = y_n^{\ell-1}(x, t_{n-1}), \quad p_n^\ell(x, t_n) &= p_{n+1}^{\ell-1}(x, t_n)
\end{aligned}
$$

In $Q_N = (0, L) \times (t_{N-1}, T)$

$$
\begin{aligned}
\partial_t y_N^\ell - \partial_{xx} y_N^\ell &= \nu^{-1} p_N^\ell \\
\partial_t p_N^\ell + \partial_{xx} p_N^\ell &= y_N^\ell - \hat{y}_N \\
y_N^\ell(x, t_{N-1}) = y_N^{\ell-1}(x, t_{N-1}), \quad p_N^\ell(x, t_N) &= 0
\end{aligned}
$$

A comment on Decompose-then-optimize

Decompose first optimization problem

$$\min_{y_n^\ell, u_n^\ell} \frac{1}{2}\|y_n^\ell - \hat{y}_n\|^2_{L^2(Q_n)} + \frac{\nu}{2}\|u_n^\ell\|^2_{L^2(Q_n)} + y_n^\ell(t_n)^\top p_{n+1}^{\ell-1}(t_n)$$

s.t.

$$\begin{aligned} \partial_t y_n^\ell - \Delta y_n^\ell &= u_n^\ell && \text{in } Q_n \\ y_n^\ell &= g && \text{on } \Sigma \\ y_n^\ell &= y_{n-1}^{\ell-1} && \text{on } \Sigma_{n-1} \end{aligned}$$

with an extra term $y_n^\ell(t_n)^\top p_{n+1}^{\ell-1}(t_n)$

A comment on Decompose-then-optimize

Decompose first optimization problem

$$\min_{y_n^\ell, u_n^\ell} \frac{1}{2}\|y_n^\ell - \hat{y}_n\|^2_{L^2(Q_n)} + \frac{\nu}{2}\|u_n^\ell\|^2_{L^2(Q_n)} + y_n^\ell(t_n)^\top p_{n+1}^{\ell-1}(t_n)$$

s.t.

$$\begin{aligned} \partial_t y_n^\ell - \Delta y_n^\ell &= u_n^\ell && \text{in } Q_n \\ y_n^\ell &= g && \text{on } \Sigma \\ y_n^\ell &= y_{n-1}^{\ell-1} && \text{on } \Sigma_{n-1} \end{aligned}$$

with an extra term $y_n^\ell(t_n)^\top p_{n+1}^{\ell-1}(t_n)$

Then optimize

$\Longrightarrow$

In $Q_1 = (0, L) \times (t_0, t_1)$

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} p_1^\ell \\ \partial_t p_1^\ell + \partial_{xx} p_1^\ell &= y_1^\ell - \hat{y}_1 \end{aligned}$$

$$y_1^\ell(x, t_0) = y_0(x), \quad p_1^\ell(x, t_1) = p_2^{\ell-1}(x, t_1)$$

In $Q_n = (0, L) \times (t_{n-1}, t_n)$

$$\begin{aligned} \partial_t y_n^\ell - \partial_{xx} y_n^\ell &= \nu^{-1} p_n^\ell \\ \partial_t p_n^\ell + \partial_{xx} p_n^\ell &= y_n^\ell - \hat{y}_n \end{aligned}$$

$$y_n^\ell(x, t_{n-1}) = y_n^{\ell-1}(x, t_{n-1}), \quad p_n^\ell(x, t_n) = p_{n+1}^{\ell-1}(x, t_n)$$

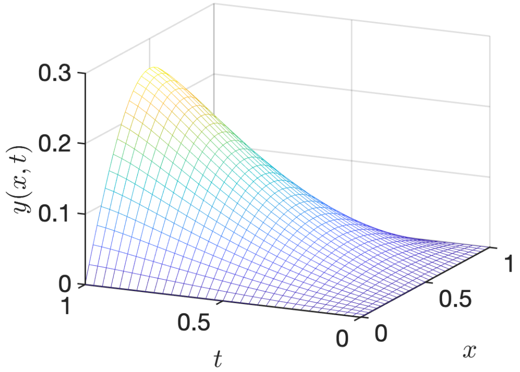
In $Q_N = (0, L) \times (t_{N-1}, T)$

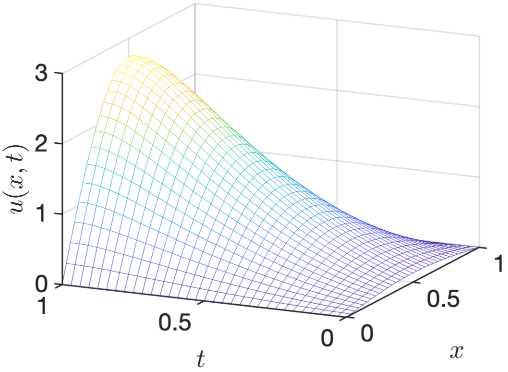
$$\begin{aligned} \partial_t y_N^\ell - \partial_{xx} y_N^\ell &= \nu^{-1} p_N^\ell \\ \partial_t p_N^\ell + \partial_{xx} p_N^\ell &= y_N^\ell - \hat{y}_N \end{aligned}$$

$$y_N^\ell(x, t_{N-1}) = y_N^{\ell-1}(x, t_{N-1}), \quad p_N^\ell(x, t_N) = 0$$

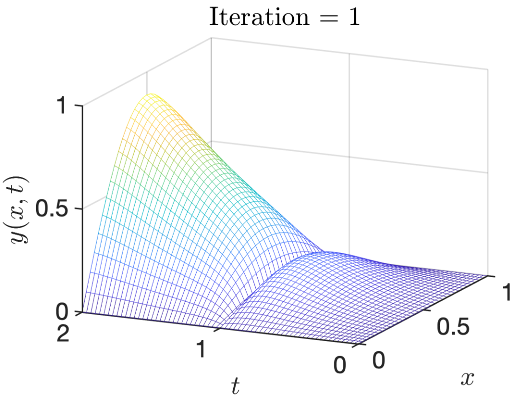
Weak scalability test

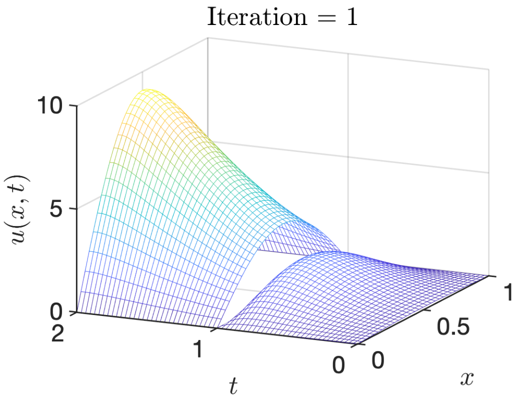
Numerical solutions: Same target $\hat{y}(x,t) = \sin(\pi x)(2t^2 + t)$ and same parameter settings



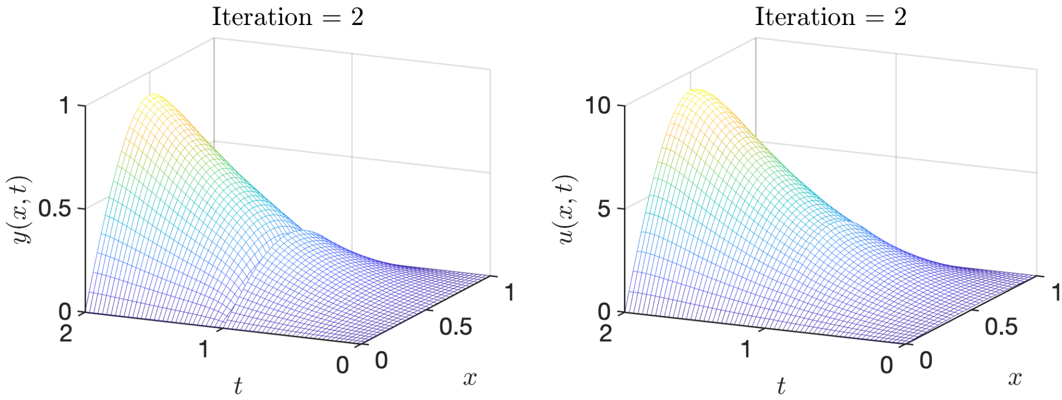


Two subdomains:



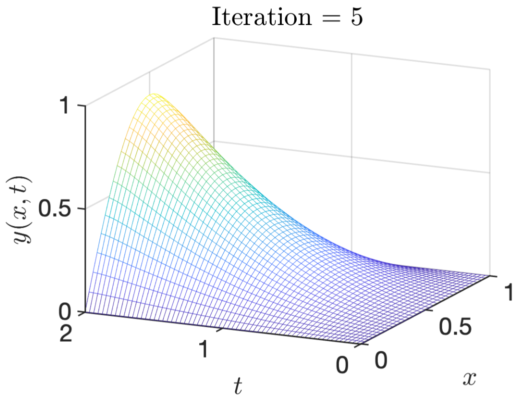


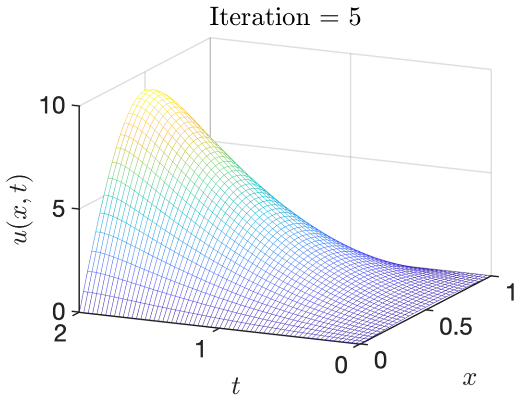
Two subdomains:



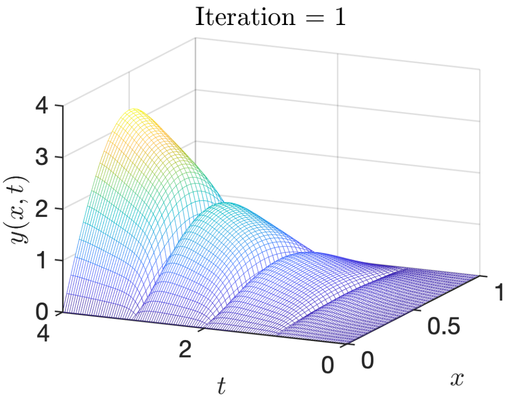
Weak scalability test

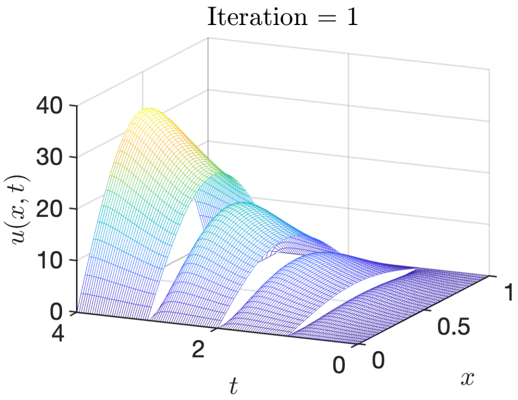
Two subdomains:





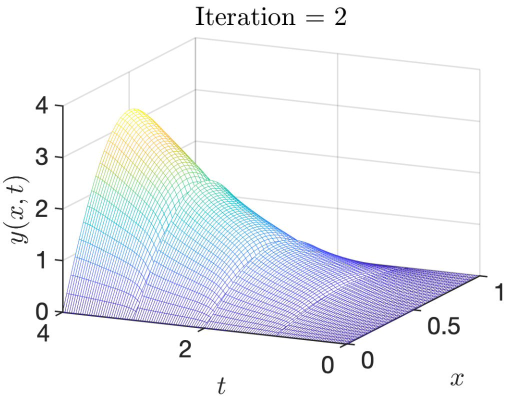
Four subdomains:

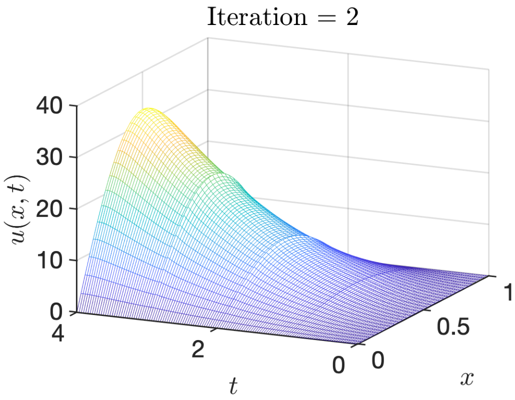




Weak scalability test

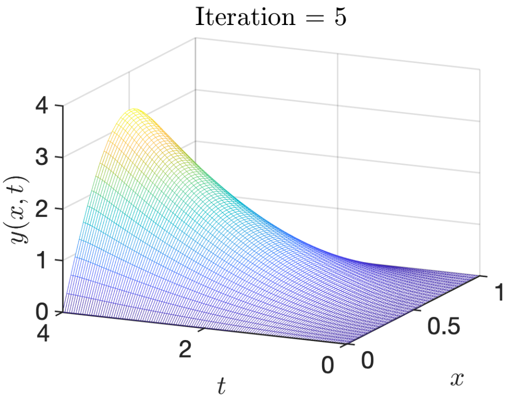
Four subdomains:

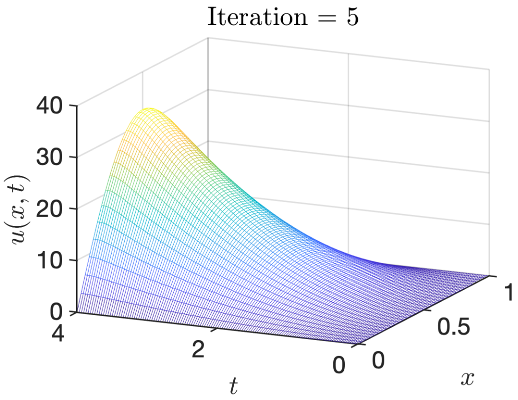




Weak scalability test

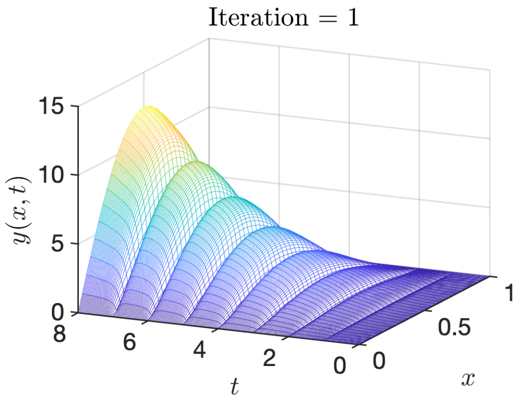
Four subdomains:

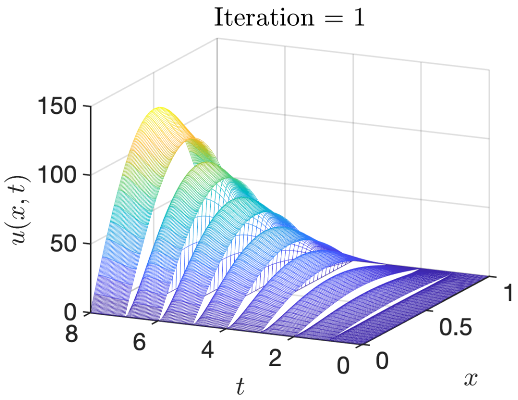




Weak scalability test

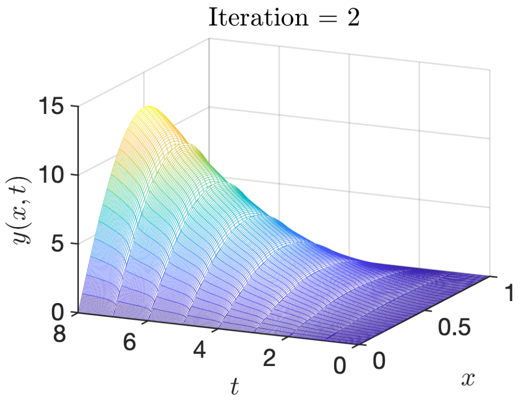
Eight subdomains:

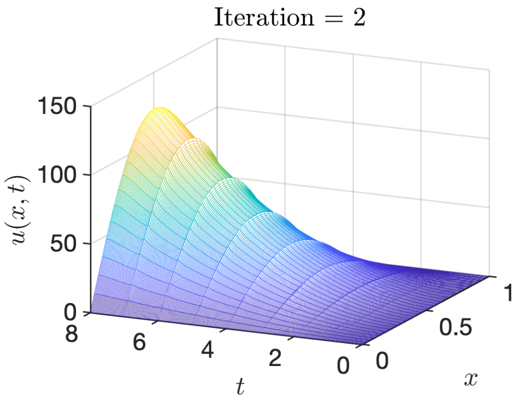




Weak scalability test

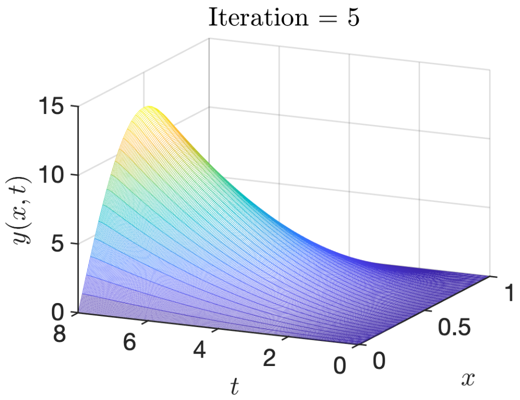
Eight subdomains:

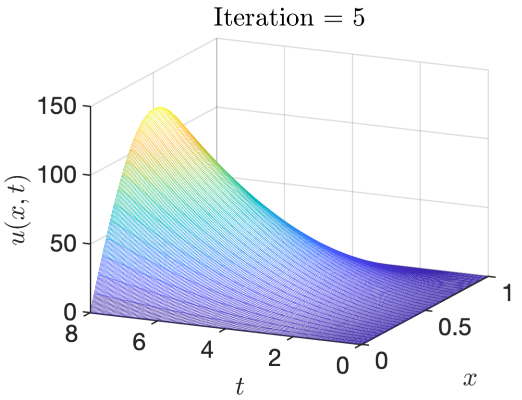




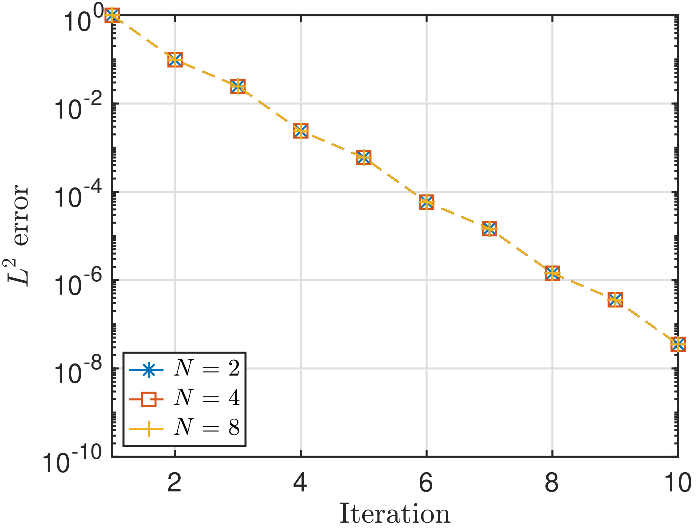
Weak scalability test

Eight subdomains:





Weak scalability test



Scalability analysis

Iteration matrix: eigendecomposition in space

$$T_{N,m} := \begin{bmatrix} T_{d,m} & T_{r,m} & & & \\ T_{l,m} & T_{d,m} & T_{r,m} & & \\ & \ddots & \ddots & \ddots & \\ & & T_{l,m} & T_{d,m} & T_{r,m} \\ & & & T_{l,m} & T_{d,m} \end{bmatrix}$$

block Toeplitz matrix with 2×2 block

$$T_{l,m} := \begin{bmatrix} 0 & 0 \\ 0 & C_{2,m}(\Delta t) \end{bmatrix}, \quad T_{r,m} := \begin{bmatrix} C_{2,m}(\Delta t) & 0 \\ 0 & 0 \end{bmatrix}$$

$$T_{d,m} := \begin{bmatrix} 0 & C_{1,m}(\Delta t) \\ -\nu C_{1,m}(\Delta t) & 0 \end{bmatrix}$$

$$C_{1,m}(\Delta t) := \frac{-\nu^{-1} \sinh(\sigma_m \Delta t)}{\sigma_m \cosh(\sigma_m \Delta t) + \lambda_m \sinh(\sigma_m \Delta t)}$$

$$C_{2,m}(\Delta t) := \frac{\sigma_m}{\sigma_m \cosh(\sigma_m \Delta t) + \lambda_m \sinh(\sigma_m \Delta t)}$$

Scalability analysis

Iteration matrix: eigendecomposition in space

$$T_{N,m} := \begin{bmatrix} T_{d,m} & T_{r,m} & & & \\ T_{l,m} & T_{d,m} & T_{r,m} & & \\ & \ddots & \ddots & \ddots & \\ & & T_{l,m} & T_{d,m} & T_{r,m} \\ & & & T_{l,m} & T_{d,m} \end{bmatrix}$$

block Toeplitz matrix with 2×2 block

$$T_{l,m} := \begin{bmatrix} 0 & 0 \\ 0 & C_{2,m}(\Delta t) \end{bmatrix}, \quad T_{r,m} := \begin{bmatrix} C_{2,m}(\Delta t) & 0 \\ 0 & 0 \end{bmatrix}$$

$$T_{d,m} := \begin{bmatrix} 0 & C_{1,m}(\Delta t) \\ -\nu C_{1,m}(\Delta t) & 0 \end{bmatrix}$$

$$C_{1,m}(\Delta t) := \frac{-\nu^{-1}\sinh(\sigma_m \Delta t)}{\sigma_m \cosh(\sigma_m \Delta t) + \lambda_m \sinh(\sigma_m \Delta t)}$$

$$C_{2,m}(\Delta t) := \frac{\sigma_m}{\sigma_m \cosh(\sigma_m \Delta t) + \lambda_m \sinh(\sigma_m \Delta t)}$$

Spectral radius of $T_{N,m}$:

$$\rho(T_{N,m}) \leq ||T_{N,m}||$$

Goal: find a **Cst** such that $||T_{N,m}|| \leq$ **Cst** < 1, and independent of N

Scalability analysis

First attempt: classical matrix norm, *e.g.*

$$||T_{N,m}||_\infty = \begin{cases} |C_{1,m}(\Delta t)| + |C_{2,m}(\Delta t)|, & \text{if } 0 < \nu \leq 1, \\ \nu|C_{1,m}(\Delta t)| + |C_{2,m}(\Delta t)|, & \text{if } \nu > 1, \end{cases}$$

Scalability analysis

First attempt: classical matrix norm, *e.g.*

$$||T_{N,m}||_\infty = \begin{cases} |C_{1,m}(\Delta t)| + |C_{2,m}(\Delta t)|, & \text{if } 0 < \nu \leq 1, \\ \nu|C_{1,m}(\Delta t)| + |C_{2,m}(\Delta t)|, & \text{if } \nu > 1, \end{cases}$$

Good news: independent of N

Bad news: larger than 1 for small ν

Observation: $\widetilde{\rho}(m) = \nu\left(C_{1,m}(\Delta t)\right)^2 + \left(C_{2,m}(\Delta t)\right)^2 < 1$

Scalability analysis

First attempt: classical matrix norm, *e.g.*

$$||T_{N,m}||_\infty = \begin{cases} |C_{1,m}(\Delta t)| + |C_{2,m}(\Delta t)|, & \text{if } 0 < \nu \leq 1, \\ \nu|C_{1,m}(\Delta t)| + |C_{2,m}(\Delta t)|, & \text{if } \nu > 1, \end{cases}$$

Remedy: a modified infinity norm

$$|||T_{N,m}||| := \max_{i=1,\dots,2(N-1)} \left(\sum_{j=1}^{2(N-1)} \left(D^{-1} T_{N,m} D\right)_{ij}^2 \right)^{\frac{1}{2}}$$

Good news: independent of N

Bad news: larger than 1 for small ν

Observation: $\widetilde{\rho}(m) = \nu \left(C_{1,m}(\Delta t)\right)^2 + \left(C_{2,m}(\Delta t)\right)^2 < 1$

$$D := \begin{bmatrix} D_d & & \\ & \ddots & \\ & & D_d \end{bmatrix}$$

$$D_d := \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\nu} \end{bmatrix}$$

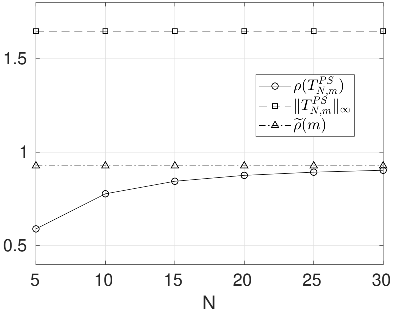
Scalability analysis

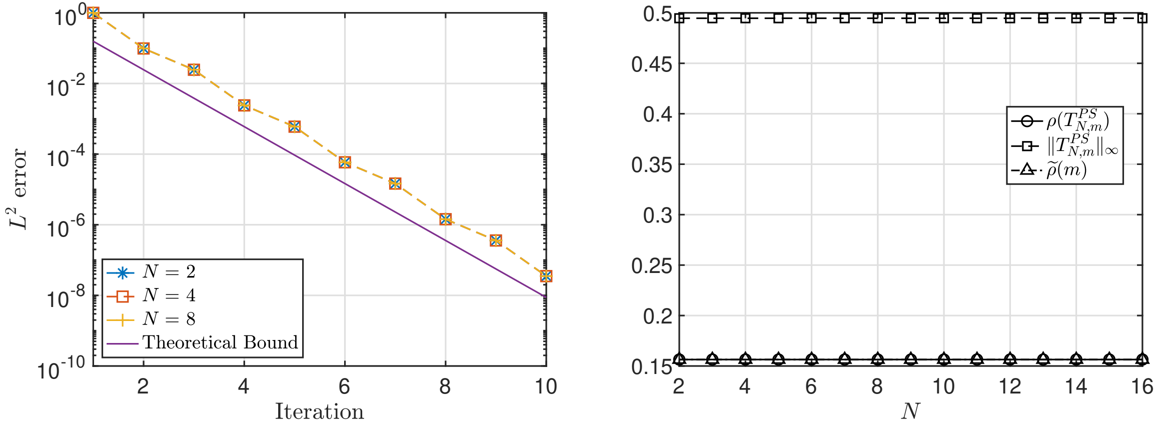
First attempt: classical matrix norm, *e.g.*

$$||T_{N,m}||_\infty = \begin{cases} |C_{1,m}(\Delta t)| + |C_{2,m}(\Delta t)|, & \text{if } 0 < \nu \leq 1, \\ \nu|C_{1,m}(\Delta t)| + |C_{2,m}(\Delta t)|, & \text{if } \nu > 1, \end{cases}$$

Remedy: a modified infinity norm

$$|||T_{N,m}||| := \max_{i=1,\ldots,2(N-1)} \left(\sum_{j=1}^{2(N-1)} \left(D^{-1} T_{N,m} D\right)_{ij}^2 \right)^{\frac{1}{2}}$$



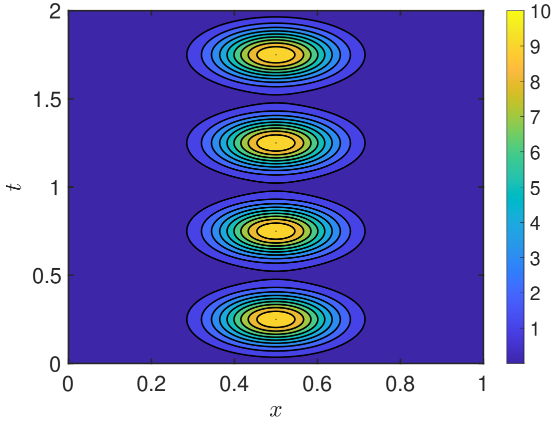


10^0
10^-2
10^-4
10^-6
10^-8
10^-10
L^2 error
2
4
6
8
10
Iteration
N = 2
N = 4
N = 8
Theoretical Bound
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
2
4
6
8
10
12
14
16
N
$\rho(T^{PS}_{N,m})$
$\|T^{PS}_{N,m}\|_\infty$
$\widetilde{\rho}(m)$

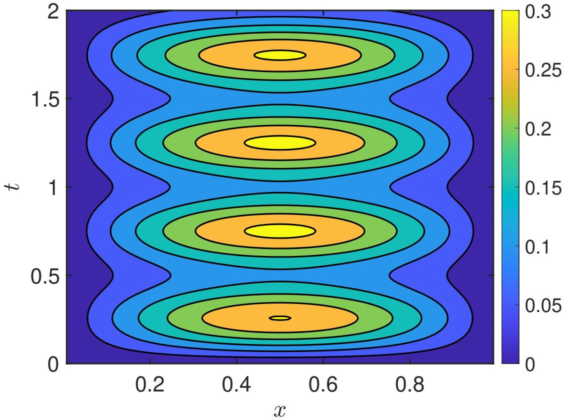
Periodic heating-cooling

Target function: $\hat{y}(x,t) = 10\sum_{n=1}^{N} \exp\left(-50\left(\left(x-\frac{L}{2}\right)^2 + \left(t-\frac{(2n-1)\Delta t}{2}\right)^2\right)\right)$

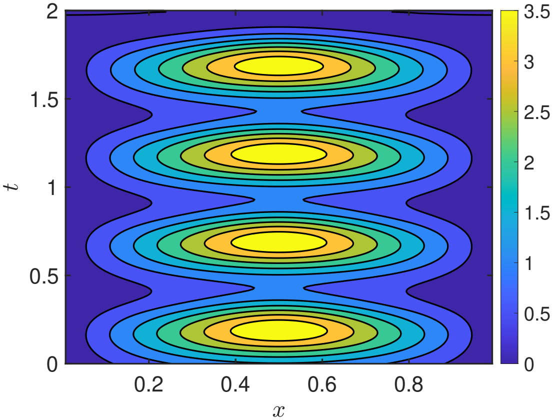
Illustration with four periods



Periodic heating-cooling

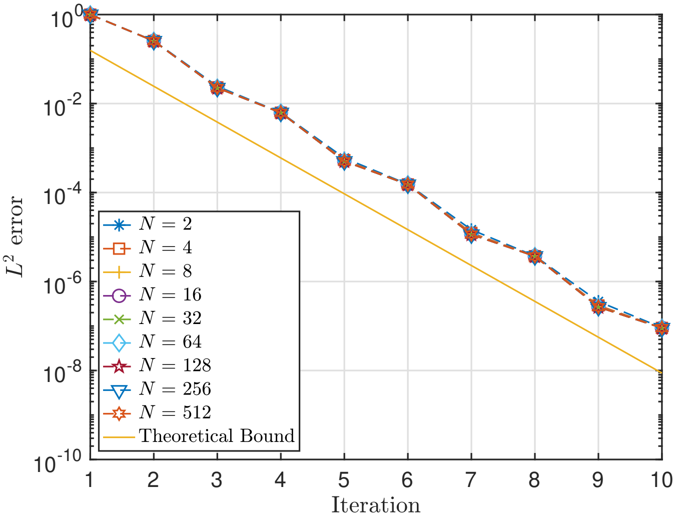


State solution y



Optimal control u

Periodic heating-cooling



Box constraint

Unconstrained

$$U_{\mathrm{ad}} = L^2(Q)$$

Optimality condition:

$$\nu u - p = 0$$

Box constraint

Unconstrained

$$U_{\mathrm{ad}} = L^2(Q)$$

Optimality condition:

$$\nu u - p = 0$$

Box constraint:

$$U_{\mathrm{ad}} := \left\{ v \in L^2(Q) : \quad u_{\min} \leq v \leq u_{\max} \right\}$$

Optimality condition:

$$u = \mathbb{P}_{U_{\mathrm{ad}}}(\nu^{-1} p) := \max\left\{ u_{\min},\, \min\{\nu^{-1} p,\, u_{\max}\} \right\}$$

Box constraint

Unconstrained

$$U_{\mathrm{ad}} = L^2(Q)$$

Box constraint:

$$U_{\mathrm{ad}} := \left\{ v \in L^2(Q) : \quad u_{\min} \leq v \leq u_{\max} \right\}$$

Optimality condition:

$$\nu u - p = 0$$

Optimality condition:

$$u = \mathbb{P}_{U_{\mathrm{ad}}}(\nu^{-1} p) := \max \left\{ u_{\min},\ \min\{\nu^{-1} p,\ u_{\max}\} \right\}$$

Reduced optimality system:

$$\begin{aligned} \partial_t y - \Delta y &= \mathbb{P}_{U_{\mathrm{ad}}}(\nu^{-1} p) && \text{in } Q & \partial_t p + \Delta p &= y - \hat{y} && \text{in } Q \\ y &= g && \text{on } \Sigma & p &= 0 && \text{on } \Sigma \\ y &= y_0 && \text{on } \Sigma_0 & p &= 0 && \text{on } \Sigma_T \end{aligned}$$

Box constraint - Two strategies

Key idea: Semismooth Newton for

$$F(u) := u - \mathbb{P}_{U_{\mathrm{ad}}}(\nu^{-1} p) = 0$$

and time domain decomposition as inner solver

Box constraint - Two strategies

Semismooth Newton step:

$$u^{k+1} = u^k + \delta u^k$$

with δu^k solves $G(u^k)[\delta u^k] = -F(u^k)$

Box constraint - Two strategies

Semismooth Newton step:

$$u^{k+1} = u^k + \delta u^k$$

with δu^k solves $G(u^k)[\delta u^k] = -F(u^k)$
Apply time domain decomposition for correction

$$\begin{aligned}
\partial_t \delta y^k - \Delta \delta y^k &= \delta u^k && \text{in } Q \\
\delta y^k &= 0 && \text{on } \Sigma \\
\delta y^k &= 0 && \text{on } \Sigma_0 \\
\partial_t \delta p^k + \Delta \delta p^k &= \delta y^k && \text{in } Q \\
\delta p^k &= 0 && \text{on } \Sigma \\
\delta p^k &= 0 && \text{on } \Sigma_T \\
\delta u^k - \chi_{I^k} \nu^{-1} \delta p^k &= -F(u^k) && \text{in } Q
\end{aligned}$$

Box constraint - Two strategies

Semismooth Newton step:

$$u^{k+1} = u^k + \delta u^k$$

with δu^k solves $G(u^k)[\delta u^k] = -F(u^k)$
Apply time domain decomposition for correction

$$\begin{aligned} \partial_t \delta y^k - \Delta \delta y^k &= \delta u^k && \text{in } Q \\ \delta y^k &= 0 && \text{on } \Sigma \\ \delta y^k &= 0 && \text{on } \Sigma_0 \\ \partial_t \delta p^k + \Delta \delta p^k &= \delta y^k && \text{in } Q \\ \delta p^k &= 0 && \text{on } \Sigma \\ \delta p^k &= 0 && \text{on } \Sigma_T \\ \delta u^k - \chi_{I^k} \nu^{-1} \delta p^k &= -F(u^k) && \text{in } Q \end{aligned}$$

Key idea: first decomposing in time into N subintervals

$$\begin{aligned} \partial_t y_n^\ell - \Delta y_n^\ell &= \mathbb{P}_{U_{\text{ad}}} \left(\nu^{-1} p_n^\ell\right) && \text{in } Q_n \\ y_n^\ell &= 0 && \text{on } \Sigma \\ y_n^\ell &= y_{n-1}^{\ell-1} && \text{on } \Sigma_{n-1} \\ \partial_t p_n^\ell + \Delta p_n^\ell &= y_n^\ell - \hat{y}_n && \text{in } Q_n \\ p_n^\ell &= 0 && \text{on } \Sigma \\ p_n^\ell &= p_{n+1}^{\ell-1} && \text{on } \Sigma_n \end{aligned}$$

Box constraint - Two strategies

Semismooth Newton step:

$$u^{k+1} = u^k + \delta u^k$$

with δu^k solves $G(u^k)[\delta u^k] = -F(u^k)$
Apply time domain decomposition for correction

$$\begin{aligned}
\partial_t \delta y^k - \Delta \delta y^k &= \delta u^k && \text{in } Q\\
\delta y^k &= 0 && \text{on } \Sigma\\
\delta y^k &= 0 && \text{on } \Sigma_0\\
\partial_t \delta p^k + \Delta \delta p^k &= \delta y^k && \text{in } Q\\
\delta p^k &= 0 && \text{on } \Sigma\\
\delta p^k &= 0 && \text{on } \Sigma_T\\
\delta u^k - \chi_{I^k} \nu^{-1} \delta p^k &= -F(u^k) && \text{in } Q
\end{aligned}$$

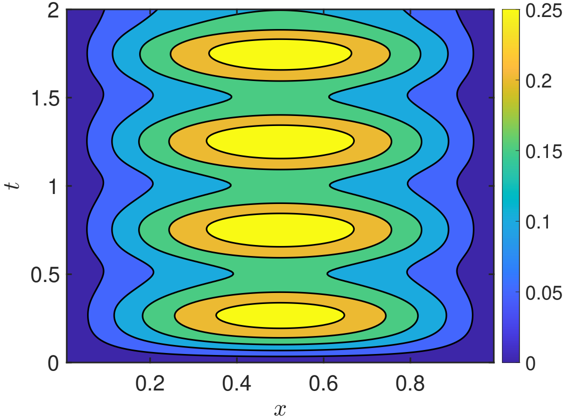
Key idea: first decomposing in time into N subintervals

$$\begin{aligned}
\partial_t y_n^\ell - \Delta y_n^\ell &= \mathbb{P}_{U_{\text{ad}}}\left(\nu^{-1} p_n^\ell\right) && \text{in } Q_n\\
y_n^\ell &= 0 && \text{on } \Sigma\\
y_n^\ell &= y_{n-1}^{\ell-1} && \text{on } \Sigma_{n-1}\\
\partial_t p_n^\ell + \Delta p_n^\ell &= y_n^\ell - \hat{y}_n && \text{in } Q_n\\
p_n^\ell &= 0 && \text{on } \Sigma\\
p_n^\ell &= p_{n+1}^{\ell-1} && \text{on } \Sigma_n
\end{aligned}$$

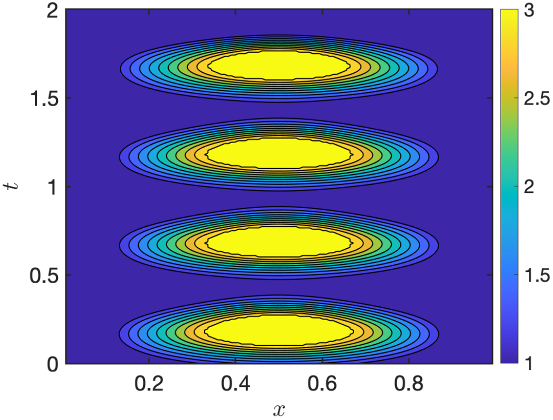
then apply semismooth Newton **within each time subinterval**.

Box constraint - Numerical results

Admissible set: $u_{\min} = 1$ and $u_{\max} = 3$



State solution y



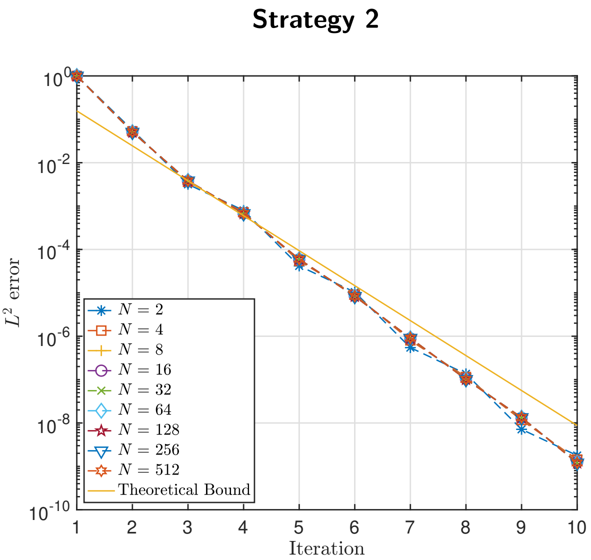
Optimal control u

Box constraint - Numerical results

Strategy 1

N	Outer SMNewton It.	Total PSM It.	Averaged PSM It.
8	3	35	11.7
16	3	46	15.3
32	3	71	23.6
64	3	77	25.7
128	3	77	25.7
256	3	77	25.7
512	3	77	25.7

Box constraint - Numerical results



What we learned

Time domain decomposition:

- Time decomposition is **different** from space decomposition
- Forward-backward structure is **important** for alternating Schwarz to converge without overlap
- Relaxation **can improve** the convergence, but is **not necessary** for the method to converge

Scalability:

- Time Parallel Schwarz is **weak** scalable (excellent scalability for heat problems)
- **Two different strategies** to handle box constraint

What we learned

Time domain decomposition:

- Time decomposition is **different** from space decomposition
- Forward-backward structure is **important** for alternating Schwarz to converge without overlap
- Relaxation **can improve** the convergence, but is **not necessary** for the method to converge

Scalability:

- Time Parallel Schwarz is **weak** scalable (excellent scalability for heat problems)
- **Two different strategies** to handle box constraint

Conclusion: Domain decomposition provides a framework rather than just a solver !

What we learned

Time domain decomposition:

- Time decomposition is **different** from space decomposition
- Forward-backward structure is **important** for alternating Schwarz to converge without overlap
- Relaxation **can improve** the convergence, but is **not necessary** for the method to converge

Scalability:

- Time Parallel Schwarz is **weak** scalable (excellent scalability for heat problems)
- **Two different strategies** to handle box constraint

Conclusion: Domain decomposition provides a framework rather than just a solver !

Thank you for your attention ! 感谢您的聆听!