Controlling the bottom topography of a microalgal pond to optimize productivity

Olivier Bernard, Liu-Di LU, Jacques Sainte-Marie, Julien Salomon

Tuesday May 25, 2020

Introduction

- Motivation: High biotechnological potential, e.g. colorants, antioxydants, cosmetics, pharmaceuticals, food complements, green energy, etc.
- Reactors

Figure: Raceways

1D Illustration

h water elevation, u horizontal averaged velocity, z_b topography. Free surface $\eta := h + z_b$, and light intensity at surface I_s.

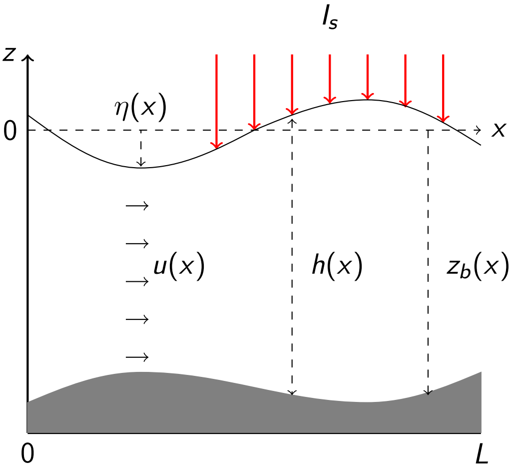

Figure: Representation of the hydrodynamic model

Shallow Water Equations

- 1D steady state shallow water equation

$$\partial_x(hu) = 0, \tag{1}$$

$$\partial_x(hu^2 + g\frac{h^2}{2}) = -gh\partial_x z_b. \tag{2}$$

Shallow Water Equations

- u, z_b as a function of h

$$u = \frac{Q_0}{h}, \tag{1}$$

$$z_b = \frac{M_0}{g} - \frac{Q_0^2}{2gh^2} - h, \tag{2}$$

$Q_0, M_0 \in \mathbb{R}^+$ are two constants.

- Froude number:

$$Fr := \frac{u}{\sqrt{gh}}$$

$Fr < 1$: subcritical case (i.e. the flow regime is fluvial)
$Fr > 1$: supercritical case (i.e. the flow regime is torrential)

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  $Fr > 1$: supercritical case (i.e. the flow regime is torrential)

- Given a smooth topography z_b, there exists a unique positive smooth solution of h which satisfies the subcritical flow condition [4, Lemma 1].

Lagrangian Trajectories

- Incompressibility of the flow: $\nabla \cdot \underline{\mathbf{u}} = 0$ with $\underline{\mathbf{u}} = (u(x), w(x,z))$

$$\partial_x u + \partial_z w = 0. \tag{3}$$

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- Integrating (3) from z_b to z and using the kinematic condition at bottom $(w(x, z_b) = u(x)\partial_x z_b)$ gives:

$$w(x,z) = (\frac{M_0}{g} - \frac{3u^2(x)}{2g} - z)u'(x).$$

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- The Beer-Lambert law describes how light is attenuated with depth:

$$I(x,z) = I_s \exp\Big(- \varepsilon(\eta(x) - z)\Big). \tag{4}$$

Han model [2]

- A: open and ready to harvest a photon,
  B: closed while processing the absorbed photon energy,
  C: inhibited if several photons have been absorbed simultaneously.
- $$\begin{cases} \dot{A} = -\sigma I A + \frac{B}{\tau}, \\ \dot{B} = \sigma I A - \frac{B}{\tau} + k_r C - k_d \sigma I B, \\ \dot{C} = -k_r C + k_d \sigma I B. \end{cases} \quad (5)$$
- A, B, C are the relative frequencies of the three possible states with $A + B + C = 1$.

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- A, B, C are the relative frequencies of the three possible states with $A + B + C = 1$.
- Following [3], (5) can be reduced to:
$$\dot{C} = -\alpha(I)C + \beta(I),$$
  where $\alpha(I) = \beta(I) + k_r$ and $\beta(I) = k_d \tau \frac{(\sigma I)^2}{\tau \sigma I + 1}$.
- The net specific growth rate:
$$\mu(C, I) := -\gamma(I)C + \zeta(I),$$
  where $\zeta(I) = \gamma(I) - R$ and $\gamma(I) = \frac{k\sigma I}{\tau \sigma I + 1}$.

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$$\bar{\mu}_\infty := \frac{1}{T}\int_0^T \frac{1}{h(x(t))}\int_{z_b(x(t))}^{\eta(x(t))} \mu(C, I(x(t), z(t)))\mathrm{d}z\mathrm{d}t,$$

$$\bar{\mu}_{N_z} := \frac{1}{T}\int_0^T \frac{1}{N_z}\sum_{i=1}^{N_z} \mu(C_i(t), I_i(x(t)))\mathrm{d}t.$$

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- Volume of the system

$$V = \int_0^L h(x)\mathrm{d}x. \tag{6}$$

- Parameterize h by a vector $a := [a_1, \cdots, a_M] \in \mathbb{R}^M$.

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- The computational chain:

$$a \to h \to u, z_b \to z_i \to I_i \to C_i \to \bar{\mu}_{N_z}.$$

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- Optimization Problem: $\bar{\mu}_{N_z}(a) := \frac{1}{T}\int_0^T \frac{1}{N_z}\sum_{i=1}^{N_z} \mu(C_i, I_i(x;a))\mathrm{d}t$, where C_i, x, z_i satisfy

$$\begin{cases} \dot{C}_i = -\alpha(I_i(x;a))C_i + \beta(I_i(x;a)) \\ \dot{x} = u(x;a) \\ \dot{z}_i = w(x, z_i; a). \end{cases}$$

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- Lagrangian

$$\mathcal{L}(C_i, z_i, x, p_{1,i}, p_{2,i}, p_3, a) := \frac{1}{T}\int_0^T \frac{1}{N_z}\sum_{i=1}^{N_z} \mu(C_i, I_i(x;a))\mathrm{d}t$$
$$- \int_0^T \sum_{i=1}^{N_z} p_{1,i}\big(\dot{C}_i + \alpha(I_i(x;a))C_i - \beta(I_i(x;a))\big)\mathrm{d}t$$
$$- \int_0^T \sum_{i=1}^{N_z} p_{2,i}\big(\dot{z}_i - w(x, z_i; a)\big)\mathrm{d}t - \int_0^T p_3\big(\dot{x} - u(x;a)\big)\mathrm{d}t.$$

Optimization Problem

The gradient $\nabla \bar{\mu}_{N_z}(a) = \partial_a \mathcal{L}$ is given by

$$\begin{aligned}\partial_a \mathcal{L} = \frac{1}{TN_z} \sum_{i=1}^{N_z} \int_0^T \big(-\gamma'(I_i(x;a)) C_i + \zeta'(I_i(x;a)) \big) \partial_a I_i(x;a) \mathrm{d}t \\ + \sum_{i=1}^{N_z} \int_0^T p_{1,i} \big(-\alpha'(I_i(x;a)) C_i + \beta'(I_i(x;a)) \big) \partial_a I_i(x;a) \mathrm{d}t \\ + \sum_{i=1}^{N_z} \int_0^T p_{2,i} \partial_a w(x, z_i; a) \mathrm{d}t + \int_0^T p_3 \partial_a u(x;a) \mathrm{d}t.\end{aligned}$$

Numerical settings

Parameterization of h: Truncated Fourier

$$h(x) = a_0 + \sum_{n=1}^{N} a_n \sin(2n\pi \frac{x}{L}). \tag{7}$$

Parameter to be optimized: Fourier coefficients $a := [a_1, \ldots, a_N]$. We use this parameterization based on the following reasons :

- We consider a hydrodynamic regime where the solutions of the shallow water equations are smooth and hence the water depth can be approximated by (7).
- One has naturally $h(0) = h(L)$ under this parameterization, which means that we have accomplished one lap of the raceway pond.
- We assume a constant volume of the system V, which can be achieved by fixing a_0. Indeed, under this parameterization and using (6), one finds $V = a_0 L$.

Convergence

We fix $N = 5$ and take 100 random initial guesses of a. For N_z varying from 1 to 80, we compute the average value of $\bar{\mu}_{N_z}$ for each N_z.

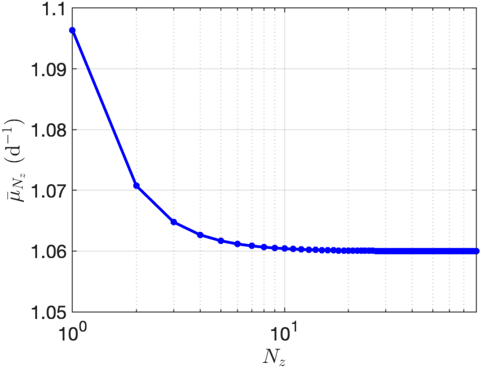

Figure: The value of $\bar{\mu}_{N_z}$ for $N_z = [1, 80]$.

Optimal Topography

We take $N_z = 40$. As an initial guess, we consider the flat topography, meaning that a is set to 0 (more details in [1]).

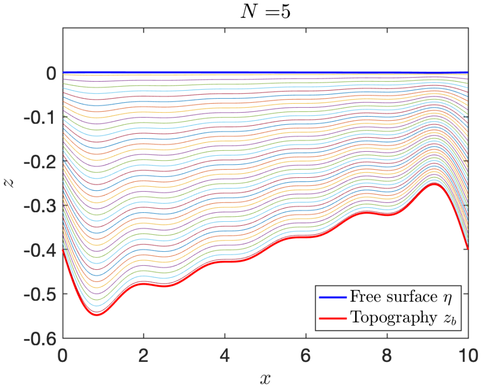

Figure: The optimal topography at the final iteration for $N = 5$. The red thick curve represents the topography (z_b), the blue thick curve represents the free surface (η).

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