Mixing Strategies Combined with Shape Design to Enhance Productivity of a Raceway Pond

Olivier Bernard, Liu-Di LU, Julien Salomon

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Introduction

- Motivation: High potential on commercial applications, e.g., cosmetics, pharmaceuticals, food complements, wastewater treatment, green energy, etc.
- Raceway ponds

Figure: A typical raceway for cultivating microalgae. Notice the paddle-wheel which mixes the culture suspension. Picture from INRA (ANR Symbiose project) [1].

1D Illustration

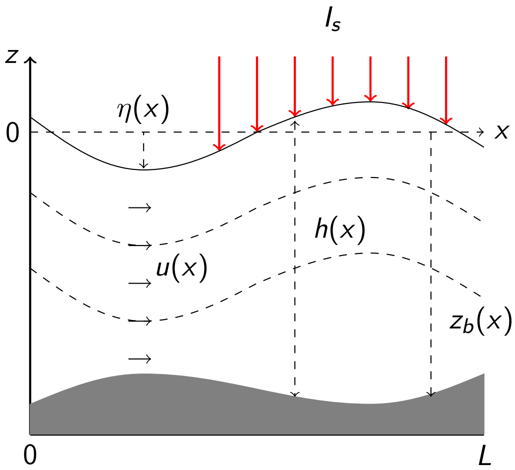

Figure: Representation of the hydrodynamic model.

Saint-Venant Equations

- 1D steady state Saint-Venant equations

$$\partial_x(hu) = 0, \quad (1)$$

$$\partial_x(hu^2 + g\frac{h^2}{2}) = -gh\partial_x z_b. \quad (2)$$

Saint-Venant Equations

- u, z_b as a function of h

$$u = \frac{Q_0}{h}, \tag{1}$$

$$z_b = \frac{M_0}{g} - \frac{Q_0^2}{2gh^2} - h, \tag{2}$$

$Q_0, M_0 \in \mathbb{R}^+$ are two constants.

- Froude number:

$$Fr := \frac{u}{\sqrt{gh}}$$

$Fr < 1$: subcritical case (i.e. the flow regime is fluvial)
$Fr > 1$: supercritical case (i.e. the flow regime is torrential)

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  $Fr < 1$: subcritical case (i.e. the flow regime is fluvial)
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- Given a smooth topography z_b, there exists a unique positive smooth solution of h which satisfies the subcritical flow condition [6, Lemma 1].

Lagrangian Trajectories

- Incompressibility of the flow: $\nabla \cdot \underline{\mathbf{u}} = 0$ with $\underline{\mathbf{u}} = (u(x), w(x,z))$

$$\partial_x u + \partial_z w = 0. \tag{3}$$

- Integrating (3) from z_b to z and using the kinematic condition at bottom ($w(x, z_b) = u(x)\partial_x z_b$) gives:

$$w(x,z) = (\frac{M_0}{g} - \frac{3u^2(x)}{2g} - z)u'(x).$$

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- A time free formulation of the Lagrangian trajectory:

$$z(x) = \color{red}{\eta(x)} + \frac{\color{red}{h(x)}}{\color{blue}{h(0)}}(\color{blue}{z(0)} - \color{blue}{\eta(0)}). \tag{4}$$

Han model [4]

- A: open and ready to harvest a photon,
  B: closed while processing the absorbed photon energy,
  C: inhibited if several photons have been absorbed simultaneously.
-
$$\begin{cases} \dot{A} = -\sigma IA + \frac{B}{\tau}, \\ \dot{B} = \sigma IA - \frac{B}{\tau} + k_r C - k_d \sigma IB, \\ \dot{C} = -k_r C + k_d \sigma IB. \end{cases} \quad (5)$$
- A, B, C are the relative frequencies of the three possible states with $A + B + C = 1$.

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- A, B, C are the relative frequencies of the three possible states with $A + B + C = 1$.
- Following [5], (5) can be reduced to:

$$\dot{C} = -(k_d \tau \frac{(\sigma I)^2}{\tau \sigma I + 1} + k_r) C + k_d \tau \frac{(\sigma I)^2}{\tau \sigma I + 1}.$$

- The net growth rate:

$$\mu(C, I) := k \sigma I A - R = k \sigma I \frac{(1 - C)}{\tau \sigma I + 1} - R,$$

- The Beer-Lambert law describes how light is attenuated with depth

$$I(x,z) = I_s \exp\Big(-\varepsilon(\eta(x) - z)\Big), \tag{6}$$

where ε is the light extinction defined by:

$$\varepsilon(X) = \alpha_0 X + \alpha_1, \tag{7}$$

with α_0 light extinction coefficient, α_1 background turbidity and X biomass concentration.

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- Constant Volume

Assumption

- the system is perfectly mixed so that biomass concentration X is homogeneous,
- the growth process occurs at a much slower time scale than those of hydrodynamics and is, as such, negligible for one lap over the raceway.

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Consequence

α_1 and X constants over the considered time scale. Hence ε is also constant and can be determined by

$$\varepsilon = \frac{1}{h} \ln(\frac{I_s}{I_{z_b}}).$$

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- Variable Volume

Assumption

Compensation condition: $\mu(I_{\bar{h}}) = 0$ [2].

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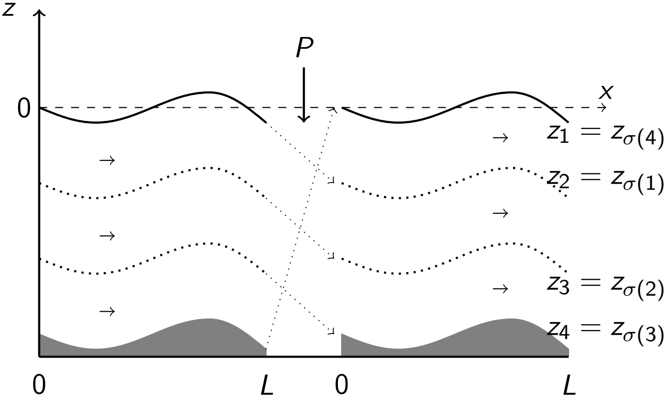
Consequence

A relation between biomass concentration (X) and average depth ($\bar{h}$)

$$X = \frac{\frac{Y_{\text{opt}}}{\bar{h}} - \alpha_1}{\alpha_0} \tag{8}$$

Mixing devices [3]

- An ideal rearrangement of trajectories: at each new lap, the algae at depth $z_i(0)$ are entirely transferred into the position $z_j(0)$ when passing through the mixing device.
- We denote by $\mathcal{P}$ the set of permutation matrices of size $N \times N$ and by $\mathfrak{S}_N$ the associated set of permutations of N elements.



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$$\bar{\mu}_\infty := \frac{1}{V}\int_0^L \int_{z_b(x)}^{\eta(x)} \mu\big(C(x,z), I(x,z)\big) \mathrm{d}z\mathrm{d}x,$$

$$\bar{\mu}_{N_z} := \frac{1}{VN_z}\sum_{i=1}^{N_z}\int_0^L \mu(C_i, I_i) h \mathrm{d}x.$$

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- Volume of the system

$$V = \int_0^L h(x)\mathrm{d}x. \tag{9}$$

- Parameterize h by a vector $a := [a_1, \cdots, a_N] \in \mathbb{R}^N$.

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- Parameterize h by a vector $a := [a_1, \cdots, a_N] \in \mathbb{R}^N$.
- The computational chain:

$$a \to h \to z_i \to I_i \to C_i \to \bar{\mu}_{N_z}.$$

- Optimization Problem:

$$\max_{P\in\mathcal{P}} \max_{a\in\mathbb{R}^N} \bar{\mu}^P_{N_z}(a) = \max_{P\in\mathcal{P}} \max_{a\in\mathbb{R}^N} \frac{1}{VN_z} \sum_{i=1}^{N_z} \int_0^L \mu(C_i^P, I_i(a))h(a)\mathrm{d}x,$$

where C_i^P satisfy

$$C_i^{P\prime} = \left(-\alpha\left(I_i(a)\right) C_i^P + \beta\left(I_i(a)\right)\right) \frac{h(a)}{Q_0},$$
$$PC^P(L) = C^P(0).$$

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- Lagrangian

$$\mathcal{L}(C_i^P, a, p_i^P) = \frac{1}{VN_z} \sum_{i=1}^{N_z} \int_0^L \Big(-\gamma(I_i(a))C_i^P + \zeta(I_i(a))\Big) h(a)\mathrm{d}x$$
$$- \sum_{i=1}^{N_z} \int_0^L p_i^P \left(C_i^{P\prime} + \frac{\alpha(I_i(a))C_i^P - \beta(I_i(a))}{Q_0} h(a)\right)\mathrm{d}x.$$

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- Lagrangian multiplier

$$p_i^{P\prime}=p_i^P\alpha(I_i(a))\frac{h(a)}{Q_0}-\frac{h(a)}{VN_z}\gamma(I_i(a)),$$
$$p^P(L)=p^P(0)P.$$

- Optimization Problem:

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- The gradient $\nabla\bar{\mu}_{N_z}(a)=\partial_a\mathcal{L}$ is given by

$$\partial_a\mathcal{L}=\sum_{i=1}^{N_z}\int_0^L\Big(\frac{-\gamma'\left(I_i\right)C_i^P+\zeta'\left(I_i\right)}{VN_z}+p_i^P\frac{-\alpha'(I_i)C_i^P+\beta'(I_i)}{Q_0}\Big)h\partial_a I_i\mathrm{d}x$$
$$+\sum_{i=1}^{N_z}\int_0^L\Big(\frac{-\gamma\left(I_i\right)C_i^P+\zeta\left(I_i\right)}{VN_z}+p_i^P\frac{-\alpha(I_i)C_i^P+\beta(I_i)}{Q_0}\Big)\partial_a h\mathrm{d}x.$$

Variable volume

- Volume related parameter a_0 as the average depth of the raceway system:

$$a_0 := \bar{h} = \frac{1}{L}\int_0^L h(x)\mathrm{d}x = \frac{V}{L}. \tag{10}$$

New parameter $\tilde{a} = [a_0, a_1, \dots, a_N]$.

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$$\Pi_{N_z}(\tilde{a}) := \bar{\mu}_{N_z}(\tilde{a})Xh(\tilde{a}) = \frac{Y_{\mathrm{opt}} - \alpha_1 a_0}{VN_z\alpha_0}\sum_{i=1}^{N_z}\int_0^L \mu(C_i^P, I_i(\tilde{a}))h(\tilde{a})\mathrm{d}x$$

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where C_i^P satisfy

$$C_i^{P'} = \left(-\alpha\left(I_i(\tilde{a})\right)C_i^P + \beta\left(I_i(\tilde{a})\right)\right)\frac{h(\tilde{a})}{Q_0},$$
$$PC^P(L) = C^P(0).$$

- Extra element in gradient: $\nabla\Pi_{N_z}(\tilde{a}) = [\partial_{a_0}\mathcal{L}, \partial_a\mathcal{L}]$.

Numerical settings

Parameterization of h: Truncated Fourier

$$h(x) = a_0 + \sum_{n=1}^{N} a_n \sin(2n\pi \frac{x}{L}). \tag{11}$$

Parameter to be optimized: Fourier coefficients $a := [a_1, \dots, a_N]$. We use this parameterization based on the following reasons :

- We consider a hydrodynamic regime where the solutions of the shallow water equations are smooth and hence the water depth can be approximated by (11).
- One has naturally $h(0) = h(L)$ under this parameterization, which means that we have accomplished one lap of the raceway pond.
- We assume a constant volume of the system V, which can be achieved by fixing a_0. Indeed, under this parameterization and using (9), one finds $V = a_0 L$.

Optimal Topography (Constant volume)

We take $N_z = 7$. As an initial guess, we consider the flat topography, meaning that a is set to 0.

$$P_{\max}^{100} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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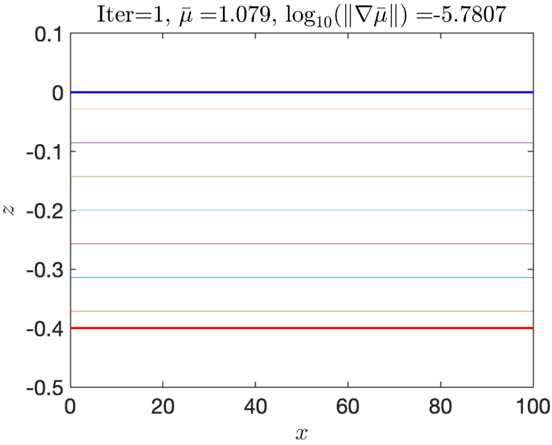

Optimal Topography (Variable volume)

We keep $N_z = 7$. As an initial guess, we consider the flat topography with $a_0 = 0.4$.

$$P_{\max}^{100} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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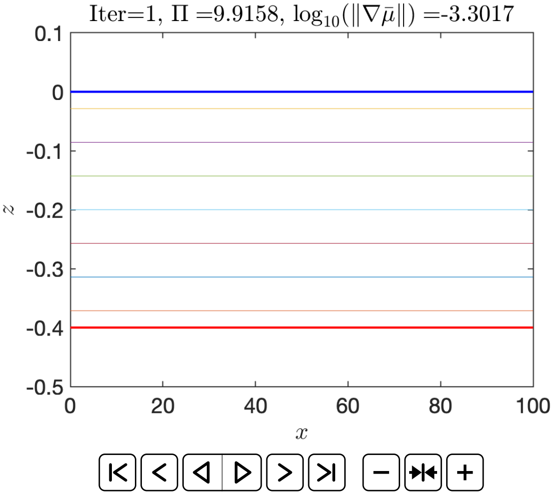

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