# Optimization problem for a microalgal raceway pond to enhance productivity

Olivier Bernard, Liu-Di LU, Jacques Sainte-Marie, Julien Salomon

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# Overview


1. Introduction
2. Raceway Modeling
3. Optimization problem
4. Numerical Experiments
5. Conclusion and Perspective

# Introduction

- Who?

# Introduction

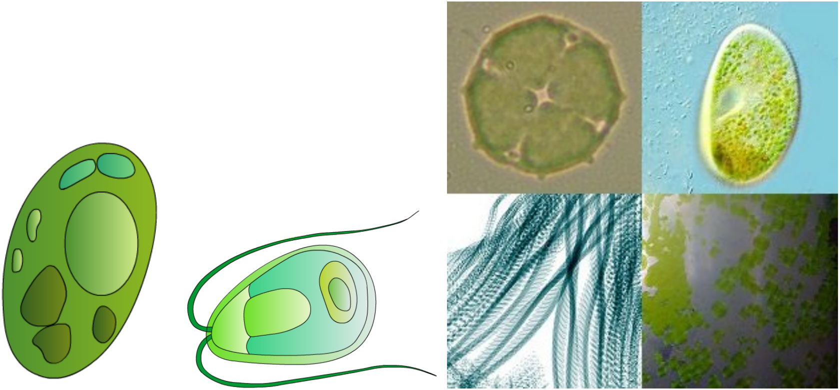

Figure: Microalgae

# Introduction

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- Why?

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- Where?

# Introduction

- Who? Microalgae: photosynthetic organisms
- Why? Biotechnological potential: colorants, antioxydants, cosmetics, pharmaceuticals, food complements, green energy, etc
- Where?
    - All aquatic environments
    - Industrial cultivation: photobioreactors

# Introduction

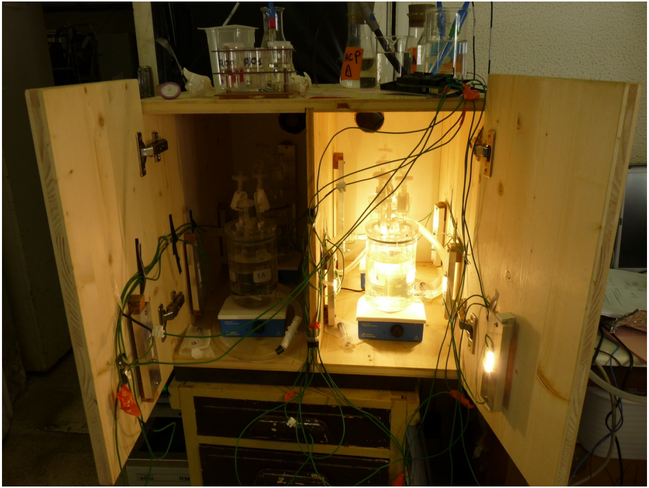

Figure: Chemostats

# Introduction

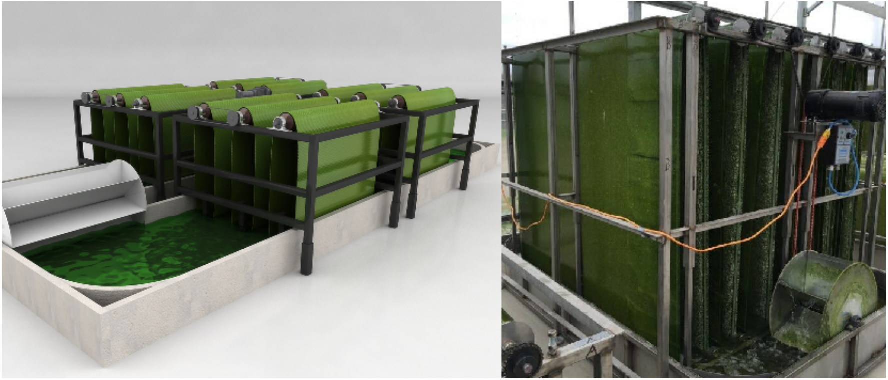

Figure: Rotating Algal Biofilm

# Introduction

Figure: Raceways

# Overview


1. Introduction
2. Raceway Modeling
   - Hydrodynamic model
   - Light intensity
   - Biologic model
   - Mixing device
3. Optimization problem
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# Shallow Water Equations

- 1D steady state shallow water equation

$$\partial_x(hu) = 0, \tag{1}$$

$$\partial_x(hu^2 + g\frac{h^2}{2}) = -gh\partial_x z_b. \tag{2}$$

- $h$ water elevation, $u$ horizontal averaged velocity, $g$ gravitational acceleration, $z_b$ topography.
- Free surface $\eta := h + z_b$, averaged discharge $Q = hu$.

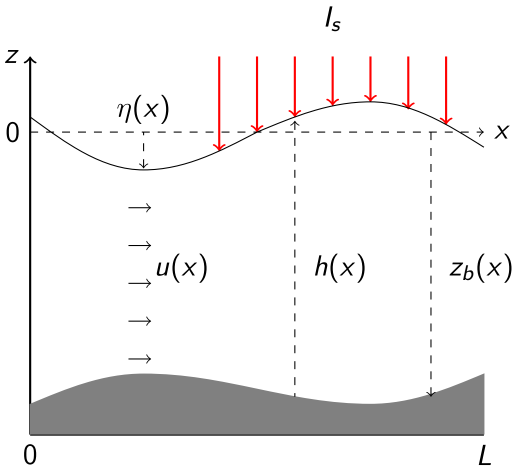


Figure: Representation of the hydrodynamic model.

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- $u, z_b$ as a function of $h$

$$u = \frac{Q_0}{h}, \tag{1}$$

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- Froude number:

$$Fr := \frac{u}{\sqrt{gh}}$$

$Fr < 1$: subcritical case (i.e. the flow regime is fluvial)
$Fr > 1$: supercritical case (i.e. the flow regime is torrential)

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  $Fr < 1$: subcritical case (i.e. the flow regime is fluvial)
  $Fr > 1$: supercritical case (i.e. the flow regime is torrential)

- Given a smooth topography $z_b$, there exists a unique positive smooth solution of $h$ which satisfies the subcritical flow condition [5, Lemma 1]

# Lagrangian Trajectories

- Incompressibility of the flow: $\nabla \cdot \underline{\mathbf{u}} = 0$ with $\underline{\mathbf{u}} = (u(x), w(x, z))$

$$\partial_x u + \partial_z w = 0. \tag{3}$$

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- Integrating (3) from $z_b$ to $z$ and using the kinematic condition at bottom $(w(x, z_b) = u(x)\partial_x z_b)$ gives:

$$w(x, z) = (\frac{M_0}{g} - \frac{3u^2(x)}{2g} - z)u'(x). \tag{4}$$

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- A time-free reformulation for $z$ as

$$z(x) = {\color{red}\eta(x)} + \frac{\color{blue}u(0)}{\color{red}u(x)}({\color{blue}z(0)} - {\color{blue}\eta(0)}), \tag{6}$$

# Beer-Lambert Law

- The Beer-Lambert law describes how light is attenuated with depth:

$$I(x, z) = I_s \exp\Big(-\varepsilon(\eta(x) - z)\Big). \tag{7}$$

Here $\varepsilon$ is the light extinction coefficient.

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- For a given position $z(0) = \eta(0) - qh(0)$ with $q \in [0,1]$, we have

$$I(x,z) = I_s \exp\Big(-\varepsilon\frac{u(0)}{u(x)}qh(0)\Big) = I_s \exp\Big(-\varepsilon q h(x)\Big)$$

# Han model

- $A$: open and ready to harvest a photon,
  $B$: closed while processing the absorbed photon energy,
  $C$: inhibited if several photons have been absorbed simultaneously.

# Han model

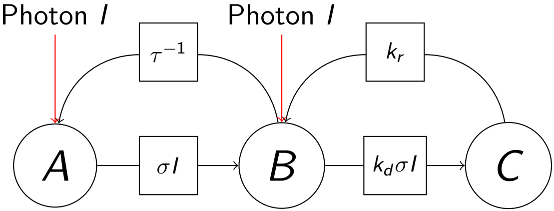


Figure: Scheme of the Han model, representing the probability of state transition, as a function of the photon flux density.

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$$\begin{cases} \dot{A} = -\sigma I A + \frac{B}{\tau}, \\ \dot{B} = \sigma I A - \frac{B}{\tau} + k_r C - k_d \sigma I B, \\ \dot{C} = -k_r C + k_d \sigma I B. \end{cases} \tag{9}$$

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- $A, B, C$ are the relative frequencies of the three possible states
  $$A + B + C = 1.$$

# Han model

- Using a fast-slow approximation and the singular perturbation theory(see [4]), this system can be reduced to one single evolution equation:

$$\dot{C} = -\alpha(I)C + \beta(I),$$

where

$$\alpha(I) = \beta(I) + k_r, \text{with } \beta(I) = k_d \tau \frac{(\sigma I)^2}{\tau \sigma I + 1}.$$

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- The net specific growth rate:

$$\mu(C, I) := -\gamma(I)C + \zeta(I),$$

where

$$\zeta(I) = \gamma(I) - R, \text{ with } \gamma(I) = \frac{k\sigma I}{\tau\sigma I + 1}.$$

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$$\bar{\mu} := \frac{1}{L}\int_0^L \frac{1}{h(x)}\int_{z_b(x)}^{\eta(x)} \mu\big(C(x,z), I(x,z)\big)\mathrm{d}z\mathrm{d}x.$$

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- In order to compute numerically, consider a uniform vertical discretization of the initial position $z(0)$ for $N_z$ cells:

$$z_i(0) = \eta(0) - \frac{i - \frac{1}{2}}{N_z} h(0), \quad i = 1, \dots, N_z.$$

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- The semi-discrete average net specific growth rate:

$$\bar{\mu}_\Delta = \frac{1}{LN_z}\sum_{i=1}^{N_z}\int_0^L \mu(C_i(x), I_i(x))\mathrm{d}x. \tag{10}$$

# Mixing device

- Paddle wheel:
  - set this hydrodynamic-biologic coupling system in motion,
  - modifies the elevation of the algae passing through it, and giving successively access to light to all the population.

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- We denote by $\mathcal{P}$ the set of permutation matrices of size $N_z \times N_z$ and by $\mathfrak{S}_{N_z}$ the associated set of permutations of $N_z$ elements.

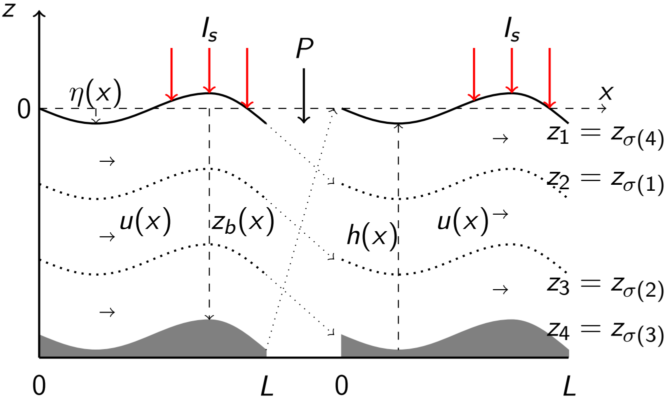


Figure: Representation of the hydrodynamic model with an example of mixing device ($P$). Here, $P$ corresponds to the cyclic permutation $\sigma = (1\ 2\ 3\ 4)$.

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**Theorem**

*The average growth rate of $K$ laps equals to one lap (see [2]).*

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# Optimization problem

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- The computational chain:

$$a \to h \to u, z_b \to z \to I \to C \to \bar{\mu}_\Delta.$$

# No permutation

- Objective function:

$$\bar{\mu}_{\Delta}(a) = \frac{1}{LN_z} \sum_{i=1}^{N_z} \int_0^L -\gamma(I_i(a))C_i + \zeta(I_i(a))\mathrm{d}x,$$

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- The optimization problem reads:
*Find $a^*$ solving the maximization problem:*

$$\max_{a \in \mathbb{R}^N} \bar{\mu}_\Delta(a).$$

# No permutation

- Lagrangian:

$$\mathcal{L}(C, p, a) = \frac{1}{LN_z}\sum_{i=1}^{N_z}\int_0^L -\gamma(I_i(a))C_i + \zeta(I_i(a))\mathrm{d}x - \sum_{i=1}^{N_z}\int_0^L p_i\big(C_i' + \frac{\alpha(I_i(a))C_i - \beta(I_i(a))}{u(a)}\big)\mathrm{d}x$$

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$$-\sum_{i=1}^{N_z}\int_0^L p_i\big(C_i' + \frac{\alpha(I_i(a))C_i - \beta(I_i(a))}{u(a)}\big)\mathrm{d}x$$

- $p_i$ the Lagrange multipliers associated with the constraint (12).

$$\begin{cases} \partial_{C_i}\mathcal{L} = p_i' - p_i\frac{\alpha(I_i(a))}{u(a)} - \frac{1}{LN_z}\gamma(I_i(a)) \\ \partial_{C_i(L)}\mathcal{L} = p_i(L). \end{cases}$$

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- If $C$ is periodic (i.e. $C(0) = C(L)$), then $\partial_{C_i(L)}\mathcal{L} = p_i(L) - p_i(0)$.

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- The gradient $\nabla \bar{\mu}_{\Delta}(a)$ is obtained by

$$\nabla \bar{\mu}_{\Delta}(a) = \partial_a \mathcal{L},$$

**Theorem**

*Under the parameterization* (11)*, if* $C$ *is periodic, then* $\nabla \bar{\mu}_{\Delta}(0) = 0$ *(see [1]).*

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- Numerically, the flat topography is the optimum.

# With a mixing device

- Objective function:

$$\bar{\mu}_{\Delta}^{P}(a) = \frac{1}{LN_z} \sum_{i=1}^{N_z} \int_0^L -\gamma(I_i(a)) C_i^P + \zeta(I_i(a)) \mathrm{d}x,$$

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- Constraint:

$$\begin{cases} C_i^{P\prime} + \frac{\alpha(I_i(a))}{u(a)} C_i^P & = \frac{\beta(I_i(a))}{u(a)} \\ P C^P(L) & = C^P(0). \end{cases} \tag{13}$$

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- Our optimization problem reads:
*Find a permutation matrix $P_{\max}$ and a parameter vector $a^*$ solving the maximization problem:*

$$\max_{P \in \mathcal{P}} \max_{a \in \mathbb{R}^M} \bar{\mu}_{\Delta}^{P}(a).$$

# With a mixing device

- Lagrangian:

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$$- \sum_{i=1}^{N_z}\int_0^L p_i^P\big(C_i^{P\prime} + \frac{\alpha(I_i(a))C_i^P - \beta(I_i(a))}{u(a)}\big)\mathrm{d}x$$

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$$-\sum_{i=1}^{N_z}\int_0^L p_i^P\left(C_i^{P\prime} + \frac{\alpha(I_i(a))C_i^P - \beta(I_i(a))}{u(a)}\right)\mathrm{d}x$$

- $p_i^P$ is the Lagrange multiplier associated with the constraint (13).

$$\begin{cases} p_i^{P\prime} - p_i^P\frac{\alpha(I_i(a))}{u(a)} - \frac{1}{LN_z}\gamma(I_i(a)) & = 0 \\ p^P(L) - p^P(0)P & = 0. \end{cases}$$

# With a mixing device

The gradient $\nabla\bar{\mu}^P_\Delta(a)$ is obtained from

$$\nabla\bar{\mu}^P_\Delta(a) = \partial_a\mathcal{L}^P,$$

where

$$\begin{aligned}\partial_a\mathcal{L}^P = \frac{1}{LN_z}\sum_{i=1}^{N_z}\int_0^L &\Big(-\gamma'(I_i(a))C_i^P + \zeta'(I_i(a))\Big)\partial_a I_i(a)\mathrm{d}x \\ &+ \sum_{i=1}^{N_z}\int_0^L p_i^P\frac{-\alpha'(I_i(a))C_i^P + \beta'(I_i(a))}{u(a)}\partial_a I_i(a)\mathrm{d}x \\ &- \sum_{i=1}^{N_z}\int_0^L p_i^P\frac{-\alpha(I_i(a))C_i^P + \beta(I_i(a))}{u^2(a)}\partial_a u(a)\mathrm{d}x.\end{aligned}$$

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- Assume that only $q$ percent of $I_s$ is available at the bottom $q \in [0, 1]$

$$\varepsilon = (1/h(0, a))\ln(1/q).$$

# Parameter Settings

- Standard settings for a raceway pond
  - Length of one lap of the raceway $L = 100\,\mathrm{m}$
  - Averaged discharge $Q_0 = 0.04\,\mathrm{m}^2 \cdot \mathrm{s}^{-1}$
  - Initial position of the topography $z_b(0) = -0.4\,\mathrm{m}$
  - First Fourier coefficient $a_0(= h(0, a)) = 0.4$
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- All the numerical parameters values for Han's model are taken from [3] and given in table 1.

Table: Parameter values for Han Model

| | | |
|---|---|---|
| $k_r$ | $6.8\ 10^{-3}$ | $\mathrm{s}^{-1}$ |
| $k_d$ | $2.99\ 10^{-4}$ | - |
| $\tau$ | $0.25$ | s |
| $\sigma$ | $0.047$ | $\mathrm{m}^2 \cdot (\mu\ \mathrm{mol})^{-1}$ |
| $k$ | $8.7\ 10^{-6}$ | - |
| $R$ | $1.389\ 10^{-7}$ | $\mathrm{s}^{-1}$ |

# Convergence of $N_z$

For 100 random *a* chosen, the average value of the functional $\bar{\mu}_\Delta$

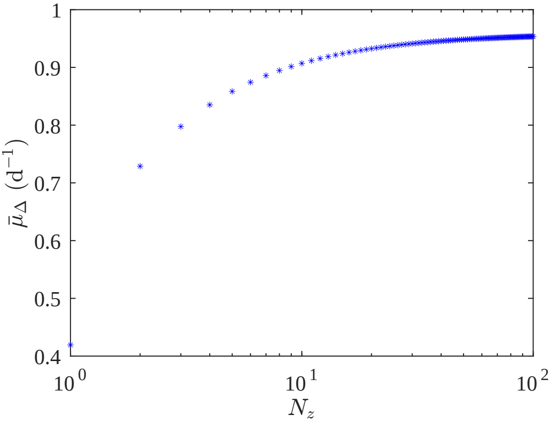


Figure: The value of the functional $\bar{\mu}_\Delta$ for $N_z = [1, 100]$.

# $C$ no periodic

The initial condition $C_0 = 0.1$

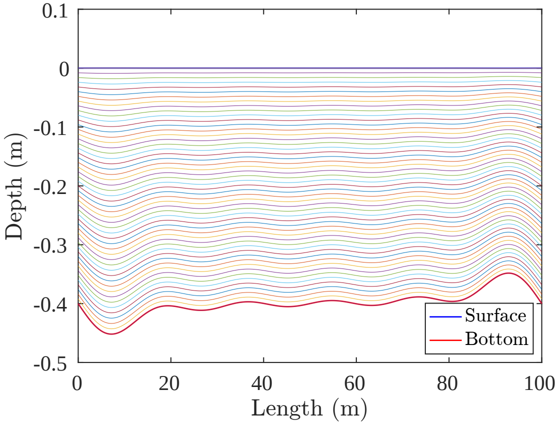


Figure: The optimal topography for $C_0 = 0.1$. The red thick line represents the topography ($z_b$), the blue thick line represents the free surface ($\eta$), and all the other curves between represent the different trajectories.

# Optimal topography for a given permutation

The permutation: $\pi = (1\ N_z)(2\ N_z - 1)(3\ N_z - 2)\cdots,$

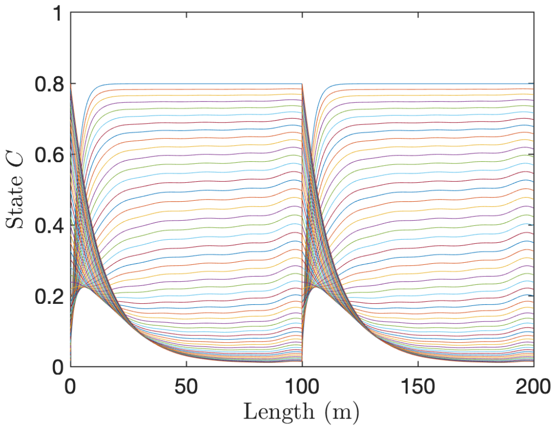


Figure: The evolution of the photo-inhibition state $C$ for two laps.

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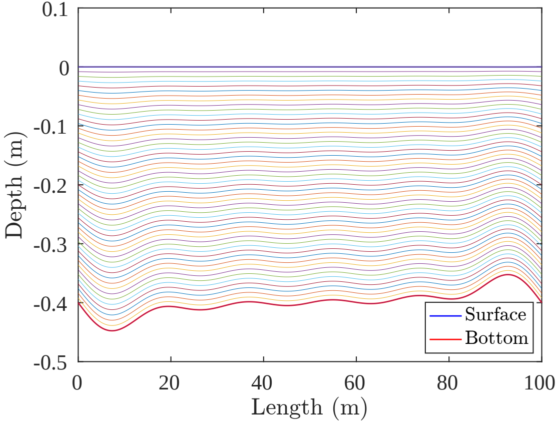


The increase in the optimal value of the objective function $\bar{\mu}_\Delta$ compared to a flat topography is around 0.228%, and compare to a flat topography and non permutation case is around 0.277%.

# Optimal matrix and optimal topography

- Test $N_z!$ cases.

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- Test $N_z!$ cases.
- Set $N_z = 7$, the optimal matrix:

$$P_{\max} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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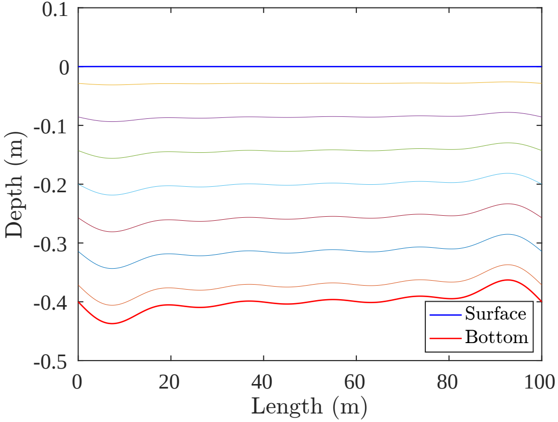


Compare to a flat topography with this $P_{\max}$, we have a gain of 0.224%, and a gain of 1.511% compare to the case a flat topography without permutation (i.e. $\mathcal{I}_{N_z}$).

# Overview


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  - An extra diffusion term in Shallow water equations or a Brownian in Lagrangian trajectories

Olivier Bernard, Liudi Lu, Jacques Sainte-Marie, and Julien Salomon.
Shape optimization of a microalgal raceway to enhance productivity.
Working paper or Preprint, November 2020.

Olivier Bernard, Liudi Lu, and Julien Salomon.
Optimizing microalgal productivity in raceway ponds through a controlled mixing device.
Working paper or Preprint, October 2020.

Jérôme Grenier, F. Lopes, Hubert Bonnefond, and Olivier Bernard.
Worldwide perspectives of rotating algal biofilm up-scaling.
Submitted, 2020.

Pierre-Olivier Lamare, Nina Aguillon, Jacques Sainte-Marie, Jérôme Grenier, Hubert Bonnefond, and Olivier Bernard.
Gradient-based optimization of a rotating algal biofilm process.
*Automatica*, 105:80–88, July 2019.

Victor Michel-Dansac, Christophe Berthon, Stéphane Clain, and Françoise Foucher.

A well-balanced scheme for the shallow-water equations with topography.
*Computers and Mathematics with Applications*, 72(3):586–593, August 2016.