Non-overlapping Domain Decomposition Methods for Elliptic Control Problems

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Optimal control

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★ **Ingredients**:

- A *system* governed by an ODE/PDE (state y),
- A *control* function u as an input to the system,
- A *target state* $\hat{y}$ as the desired state of the system,
- A *cost functional* J, e.g., cost of u, discrepancy between y and $\hat{y}$, etc.

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★ **Goal**:

- Find the control $u^\star$ which minimizes the cost such that the system reaches the desired state.

Example 1

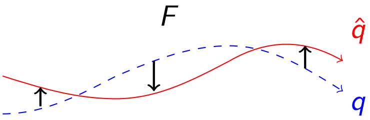
Problem: Compute the force of thrust F

$$\min_{F\in U_{\mathrm{ad}}} \frac{1}{2}\|F\|^2_{U_{\mathrm{ad}}} + \frac{1}{2}\int_0^T |q(t)-\hat{q}(t)|^2 \mathrm{d}t,$$

subject to

$$\ddot{q} = -\frac{q}{|q|^3} + \frac{F}{m}, \qquad \text{in } (0, T),$$

with m the mass of the satellite.



Example 2

Problem: Compute the bottom topography z_b

$$\max_{z_b \in U_{\text{ad}}} \mathcal{P}(z_b, X, I),$$

subject to

$$\dot{X} = f(X, I),$$

with I the light perceived.

Example 3

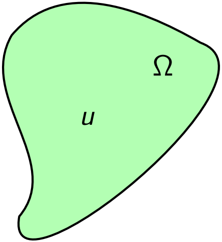
Problem: Compute the heat source u

$$\min_{u \in U_{\mathrm{ad}}} \frac{1}{2}\|u\|_{U_{\mathrm{ad}}}^2 + \frac{1}{2}\int_\Omega |y(x) - \hat{y}(x)|^2 \mathrm{d}x,$$

subject to

$$-\mathrm{div}\left(\kappa(x)\nabla y(x)\right) = u(x), \quad \text{in } \Omega,$$

$\kappa(x)$ the thermal conductivity of Ω.



Poisson's equation

★ **Model**:

$$\begin{aligned} -\Delta y &= u \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

with $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$.

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$$J(y, u) = \frac{1}{2} \int_\Omega |y(x) - \hat{y}(x)|^2 \, \mathrm{d}x + \frac{\nu}{2} \|u\|^2_{U_{\text{ad}}}, \tag{2}$$

with $\nu > 0$.

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subject to the PDE constraint (1).

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★ Lagrange multiplier approach:

$$
\mathcal{L}(y, \lambda, u) = J(y, u) + \langle \lambda, -\Delta y - u \rangle,
$$

λ is the Lagrange multiplier or adjoint state.

Optimality system

$$\mathcal{L}(y, \lambda, u) = \frac{1}{2}\int_{\Omega} |y(x) - \hat{y}(x)|^2 \,\mathrm{d}x + \frac{\nu}{2}\|u\|^2_{U_{\text{ad}}} + \langle \lambda, -\Delta y - u \rangle$$

- Primal problem:

$$\partial_\lambda \mathcal{L}(y, \lambda, u) = 0, \quad \Rightarrow \quad -\Delta y - u = 0.$$

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- Integration by parts

$$\langle \lambda, -\Delta y \rangle = \langle -\Delta \lambda, y \rangle - \int_{\partial\Omega} \lambda \partial_n y + \int_{\partial\Omega} y \partial_n \lambda.$$

Optimality system

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- Optimality condition:

$$\partial_u L(y,\lambda,u) = 0 \quad \Rightarrow \quad -\lambda + \nu u = 0.$$

with $U_{\mathrm{ad}} := L^2(\Omega)$.

Optimality system

★ First-order optimality system:

$$-\Delta y = u \text{ in } \Omega, \qquad -\Delta\lambda = y - \hat{y} \quad \text{in } \Omega,$$
$$y = 0 \text{ on } \partial\Omega, \qquad \lambda = 0 \qquad \text{on } \partial\Omega,$$

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★ First-order optimality system:

$$\nu \Delta^2 y = y - \hat{y} \quad \text{in } \Omega,$$
$$y = 0 \quad \text{on } \partial\Omega,$$
$$\Delta y = 0 \quad \text{on } \partial\Omega.$$

Dirichlet-Neumann (Bjørstad, Widlund 1986)

One dimensional case: $\Omega = (0,1)$, $\Omega_1 = (0,\Gamma), \Omega_2 = (\Gamma, 1)$ with Γ the interface, $y_1^k(0) = y_2^k(1) = D^{(2)}y_1^k(0) = D^{(2)}y_2^k(1) = 0$, and $\hat{y} = 0$.

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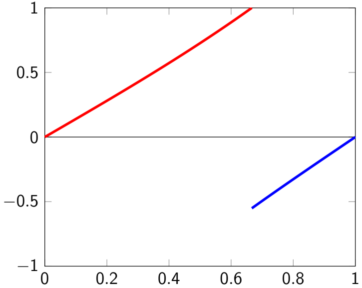

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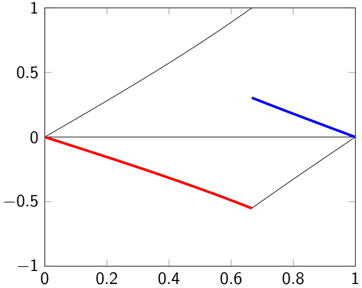

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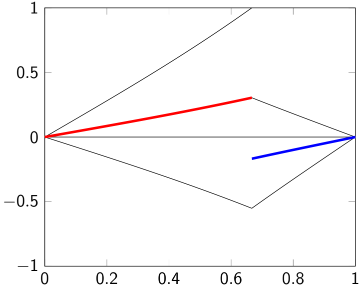

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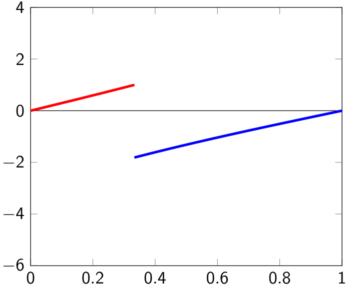

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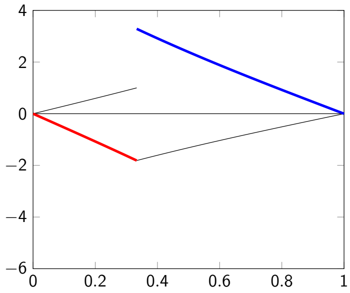

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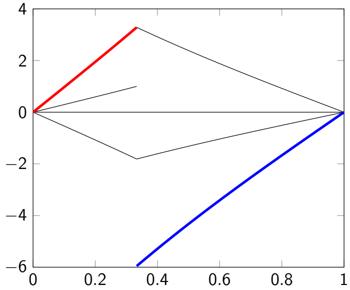
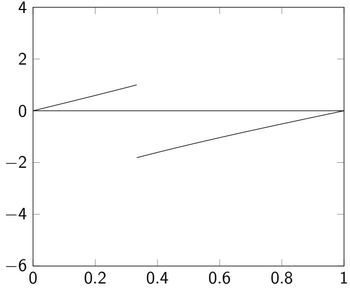


Dirichlet-Neumann with relaxation

Relaxation parameter: $\theta_1, \theta_2 \in (0,1)$.

$$\nu D^{(4)} y_1^k - y_1^k = 0, \qquad \nu D^{(4)} y_2^k - y_2^k = 0,$$
$$y_1^k(\Gamma) = y_\Gamma^{k-1}, \qquad D^{(1)} y_2^k(\Gamma) = D^{(1)} y_1^k(\Gamma),$$
$$D^{(2)} y_1^k(\Gamma) = \tilde{y}_\Gamma^{k-1}, \qquad D^{(3)} y_2^k(\Gamma) = D^{(3)} y_1^k(\Gamma),$$

$y_\Gamma^k := \theta_1 y_2^{k-1}(\Gamma) + (1-\theta_1) y_\Gamma^{k-1}$, $\tilde{y}_\Gamma^k := \theta_2 D^{(2)} y_2^{k-1}(\Gamma) + (1-\theta_2)\tilde{y}_\Gamma^{k-1}$.



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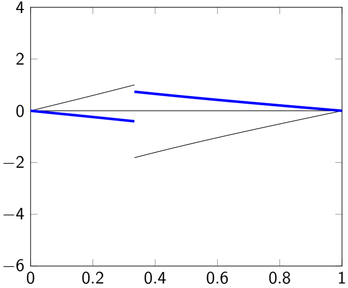

Convergence Analysis

- Solve y_1^k, y_2^k by using $y_1^k(0) = y_2^k(1) = D^{(2)}y_1^k(0) = D^{(2)}y_2^k(1) = 0$.

$$
\begin{aligned}
y_1^k(x) &= A^k \sinh(\frac{\mu x}{\sqrt{2}}) \cos(\frac{\mu x}{\sqrt{2}}) + B^k \cosh(\frac{\mu x}{\sqrt{2}}) \sin(\frac{\mu x}{\sqrt{2}}), \\
y_2^k(x) &= C^k \sinh(\frac{\mu(1-x)}{\sqrt{2}}) \cos(\frac{\mu(1-x)}{\sqrt{2}}) \\
&\qquad + E^k \cosh(\frac{\mu(1-x)}{\sqrt{2}}) \sin(\frac{\mu(1-x)}{\sqrt{2}}).
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- Evaluate A^k, B^k, C^k, E^k by using the transmission condition at Γ.

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- Evaluate A^k, B^k, C^k, E^k by using the transmission condition at Γ.
- Different behaviours according to the interface Γ:

Gander, Kwok and Mandal, *Convergence of Substructuring Methods for Elliptic Optimal Control Problems*, 2018

Neumann-Neumann (Bourgat, Glowinski, Tallec, Vidrascu 1989)

Primal problem:

$$\begin{aligned} -\Delta y_j^k &= \nu^{-1}\lambda_j^k, && \text{in } \Omega_j & -\Delta \psi_j^k &= 0, && \text{in } \Omega_j \\ y_j^k &= 0, && \text{on } \partial\Omega_j/\ \Gamma & \psi_j^k &= 0, && \text{on } \partial\Omega_j/\ \Gamma \\ y_j^k &= y_\Gamma^{k-1}, && \text{on } \Gamma & \partial_{n_j}\psi_j^k &= \partial_{n_1} y_1^k + \partial_{n_2} y_2^k, && \text{on } \Gamma \end{aligned}$$

with $y_\Gamma^k := y_\Gamma^{k-1} - \theta_1 \left(\psi_1^k|_\Gamma + \psi_2^k|_\Gamma\right)$.

Adjoint problem:

$$\begin{aligned} -\Delta \lambda_j^k &= y_j^k - \hat{y}, && \text{in } \Omega_j & -\Delta \phi_j^k &= 0, && \text{in } \Omega_j \\ \lambda_j^k &= 0, && \text{on } \partial\Omega_j/\ \Gamma & \phi_j^k &= 0, && \text{on } \partial\Omega_j/\ \Gamma \\ \lambda_j^k &= \lambda_\Gamma^{k-1}, && \text{on } \Gamma & \partial_{n_j}\phi_j^k &= \partial_{n_1} \lambda_1^k + \partial_{n_2} \lambda_2^k, && \text{on } \Gamma \end{aligned}$$

with $\lambda_\Gamma^k := \lambda_\Gamma^{k-1} - \theta_2 \left(\phi_1^k|_\Gamma + \phi_2^k|_\Gamma\right)$.

Optimality system

$$\mathcal{L}(y,\lambda,u) = \frac{1}{2}\int_\Omega |y(x)-\hat{y}(x)|^2\,\mathrm{d}x + \frac{\nu}{2}\|u\|^2_{U_{\mathrm{ad}}} + \langle \lambda, -\Delta y - u\rangle$$

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- Optimality condition:

$$\partial_u L(y,\lambda,u) = 0,$$

with $U_{\mathrm{ad}} := H^{-1}(\Omega)$.

Energy norm

- A linear operator $\mathcal{H} : H^{-1}(\Omega) \to H_0^1(\Omega) \subset L^2(\Omega)$ such that $\mathcal{H}u$ is the unique solution of the variational problem related to (1)

$$\int_\Omega \nabla \mathcal{H}u(x) \cdot \nabla v(x) \, \mathrm{d}x = \langle u, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

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- The norm in $H^{-1}(\Omega)$ which is equivalent to the energy norm

$$\|u\|_{H^{-1}(\Omega)}^2 := \langle u, \mathcal{H}u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \|\nabla y\|_{L^2(Q)}^2.$$

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Optimality system

$$\mathcal{L}(y,\lambda,u) = \frac{1}{2}\int_\Omega |y(x)-\hat{y}(x)|^2 \,\mathrm{d}x + \frac{\nu}{2}\langle \mathcal{H}u, u\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} + \langle \lambda, -\Delta y - u\rangle.$$

- Primal problem:

$$\partial_\lambda L(y,\lambda,u) = 0, \quad \Rightarrow \quad \begin{aligned} -\Delta y &= u \quad \text{in } \Omega,\\ y &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

- Adjoint problem:

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- Optimality condition:

$$\partial_u L(y,\lambda,u) = 0 \quad \Rightarrow \quad -\lambda + \nu\mathcal{H}u = 0.$$

Optimality system

★ First-order optimality system:

$$\begin{aligned} -\Delta y &= u \text{ in } \Omega, & -\Delta \lambda &= y - \hat{y} \quad \text{in } \Omega, \\ y &= 0 \text{ on } \partial\Omega, & \lambda &= 0 \qquad\quad \text{on } \partial\Omega, \end{aligned}$$

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$$\begin{aligned} -\nu \Delta y + y &= \hat{y} \text{ in } \Omega,\\ y &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Error Analysis (DN)

One dimensional case: $\Omega = (0,1)$, $\Omega_1 = (0,\Gamma), \Omega_2 = (\Gamma,1)$ with Γ the interface.

- Error equation for $e_j^k := y - y_j^k$

$$\begin{aligned} \nu \ddot{e}_1^k - e_1^k = 0, \quad e_1^k(0) = 0, \quad e_1^k(\Gamma) = e_2^{k-1}(\Gamma), \\ \nu \ddot{e}_2^k - e_2^k = 0, \quad e_2^k(1) = 0, \quad \dot{e}_2^k(\Gamma) = \dot{e}_1^k(\Gamma). \end{aligned}$$

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- Solution:

$$e_1^k(x) = A^k \sinh(\sqrt{\nu^{-1}}x), \quad e_2^k(x) = B^k \sinh\left(\sqrt{\nu^{-1}}(1-x)\right) e^{-\sqrt{\nu^{-1}}}.$$

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- Convergence factor:

$$e_2^k(\Gamma) = -e_2^{k-1}(\Gamma) \underbrace{\tanh\left(\sqrt{\nu^{-1}}(1-\Gamma)\right) \coth\left(\sqrt{\nu^{-1}}\Gamma\right)}_{\rho_{\mathrm{DN}}}.$$

Error Analysis (DNR)

- Error equation for $e_j^k := y - y_j^k$

$$\nu \ddot{e}_1^k - e_1^k = 0, \quad e_1^k(0) = 0, \quad e_1^k(\Gamma) = e_\Gamma^{k-1},$$
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with $e_\Gamma^k := (1-\theta)e_\Gamma^{k-1} + \theta e_2^k(\Gamma)$, $\theta \in (0,1)$.

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with $e_\Gamma^k := (1-\theta)e_\Gamma^{k-1} + \theta e_2^k(\Gamma)$, $\theta \in (0,1)$.

- Convergence factor:

$$\rho_{\text{DNR}} := \left| 1 - \theta \left[1 + \tanh\left(\sqrt{\nu^{-1}}(1-\Gamma)\right) \coth\left(\sqrt{\nu^{-1}}\Gamma\right) \right] \right|.$$

Error Analysis (NN)

- Error equation for $e_j^k := y - y_j^k$:

$$\nu\ddot{e}_j^k - e_j^k = 0, \quad e_1^k(0) = 0, \quad e_2^k(1) = 0, \quad e_j^k(\Gamma) = e_\Gamma^{k-1},$$
$$\nu\ddot{\psi}_j^k - \psi_j^k = 0, \quad \psi_1^k(0) = 0, \quad \psi_2^k(1) = 0, \quad \partial_{n_j}\psi_j^k = \partial_{n_1}e_1^k + \partial_{n_2}e_2^k,$$

with $e_\Gamma^k := e_\Gamma^{k-1} - \theta\left(\psi_1^k(\Gamma) + \psi_2^k(\Gamma)\right)$, $\theta \in (0,1)$.

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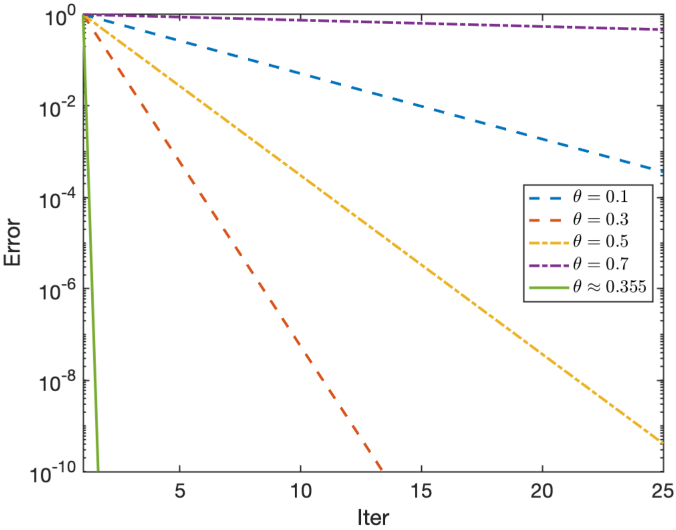
with $e_\Gamma^k := e_\Gamma^{k-1} - \theta\left(\psi_1^k(\Gamma) + \psi_2^k(\Gamma)\right)$, $\theta \in (0,1)$.

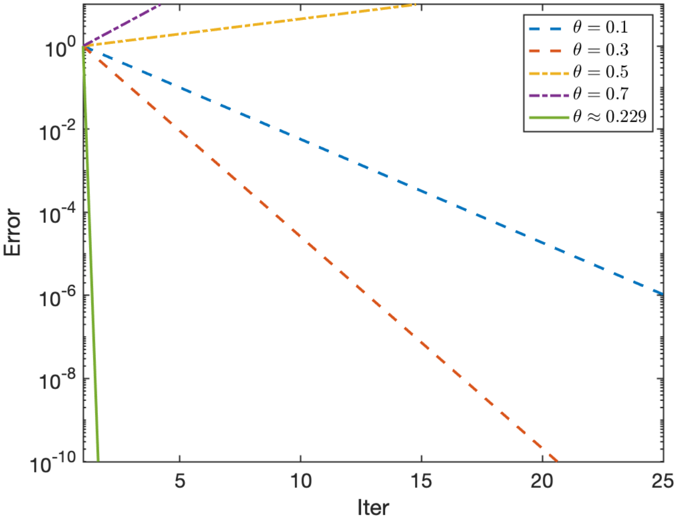
- Convergence factor:

$$\rho_{\text{NN}} := \left|1 - \theta\Big(\tanh(\sqrt{\nu^{-1}}\Gamma) + \tanh(\sqrt{\nu^{-1}}(1-\Gamma))\Big)\Big(\coth(\sqrt{\nu^{-1}}\Gamma) + \coth(\sqrt{\nu^{-1}}(1-\Gamma))\Big)\right|.$$

Convergence tests (DNR)



Convergence tests (NN)



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Thanks for your attention !