Dirichlet-Neumann and Neumann-Neumann Methods for Elliptic Control Problems

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Gander, Kwok and Mandal, *Convergence of Substructuring Methods for Elliptic Optimal Control Problems*, 2018

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What else?

- Optimal control under H^{-1} regularization
  - Langer, Steinbach, Tröltzsch and Yang, *Space-time finite element discretization of parabolic optimal control problems with energy regularization*, 2021
  - Neumüller and Steinbach, *Regularization error estimates for distributed control problems in energy spaces*, 2021
- Cost functional:

$$J(y,u) = \frac{1}{2}\|y - \hat{y}\|^2_{L^2(Q)} + \frac{\nu}{2}\|u\|^2_{U_{ad}}.$$

- *... we obtain a singularly perturbed Dirichlet boundary value problem for the Poisson equation, while for the control in* $L^2(\Omega)$*, this is a singularly perturbed problem for the BiLaplace operator.*

Poisson's equation

★ **Model**:

$$\begin{aligned} -\Delta y &= u \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

with $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$.

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with $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$.

★ **Problem**:

$$J(y, u) = \frac{1}{2} \int_{\Omega} |y(x) - \hat{y}(x)|^2 \, \mathrm{d}x + \frac{\nu}{2} \|u\|^2_{U_{\text{ad}}}, \tag{2}$$

with $\nu > 0$.

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with $\nu > 0$.

★ **Goal**: Find

$$\min_{u \in U_{\text{ad}}} J(y,u),$$

subject to the PDE constraint (1).

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subject to the PDE constraint (1).

★ Lagrange multiplier approach:

$$\mathcal{L}(y,\lambda,u) = J(y,u) - \langle \lambda, -\Delta y - u \rangle,$$

λ is the Lagrange multiplier or adjoint state.

Optimality system

★ First-order optimality system:

$$
\begin{aligned}
-\Delta y &= u \text{ in } \Omega, & -\Delta \lambda &= y - \hat{y} \quad \text{in } \Omega,\\
y &= 0 \text{ on } \partial\Omega, & \lambda &= 0 \qquad\quad \text{on } \partial\Omega,
\end{aligned}
$$

$$
-\lambda + \nu u = 0.
$$

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$$-\Delta y = \nu^{-1}\lambda \quad \text{in } \Omega, \qquad -\Delta\lambda = y - \hat{y} \quad \text{in } \Omega,$$
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★ First-order optimality system:

$$\nu\Delta^2 y = y - \hat{y} \quad \text{in } \Omega,$$
$$y = 0 \quad \text{on } \partial\Omega,$$
$$\Delta y = 0 \quad \text{on } \partial\Omega.$$

Dirichlet-Neumann (Bjørstad, Widlund 1986)

Example: $\Omega_1 = (0, \Gamma), \Omega_2 = (\Gamma, 1)$ with $\Gamma = \frac{1}{2}$ the interface, $y_1^k(0) = y_2^k(1) = D^{(2)} y_1^k(0) = D^{(2)} y_2^k(1) = 0$, and $\hat{y} = 0$.

$$
\begin{aligned}
\nu D^{(4)} y_1^k - y_1^k &= 0, & \nu D^{(4)} y_2^k - y_2^k &= 0,\\
y_1^k(\Gamma) &= y_2^{k-1}(\Gamma), & D^{(1)} y_2^k(\Gamma) &= D^{(1)} y_1^k(\Gamma),\\
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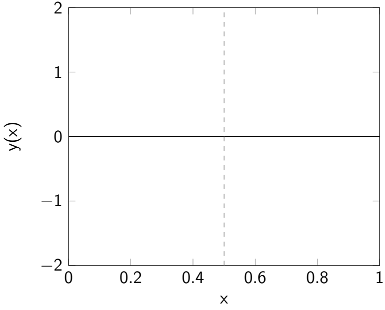
$$



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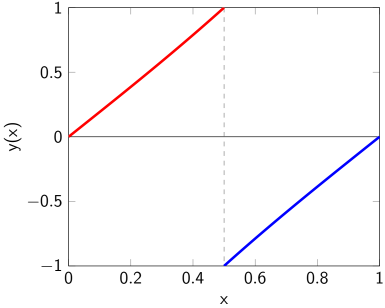
$$\begin{aligned} \nu D^{(4)} y_1^k - y_1^k &= 0, & \nu D^{(4)} y_2^k - y_2^k &= 0, \\ y_1^k k(\Gamma) &= y_2^{k-1}(\Gamma), & D^{(1)} y_2^k(\Gamma) &= D^{(1)} y_1^k(\Gamma), \\ D^{(2)} y_1^k(\Gamma) &= D^{(2)} y_2^{k-1}(\Gamma), & D^{(3)} y_2^k(\Gamma) &= D^{(3)} y_1^k(\Gamma). \end{aligned}$$



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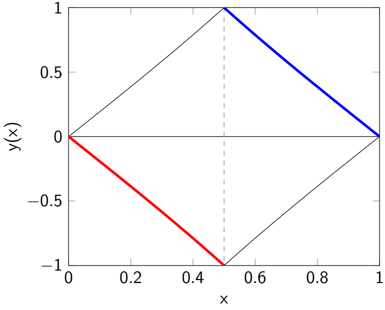

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$$\nu D^{(4)} y_1^k - y_1^k = 0, \qquad \nu D^{(4)} y_2^k - y_2^k = 0,$$
$$y_1^k k(\Gamma) = y_2^{k-1}(\Gamma), \qquad D^{(1)} y_2^k(\Gamma) = D^{(1)} y_1^k(\Gamma),$$
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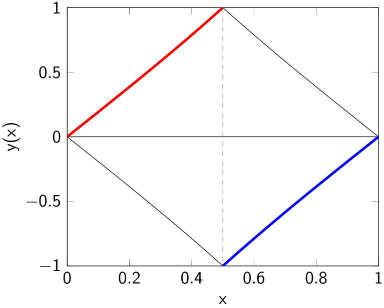

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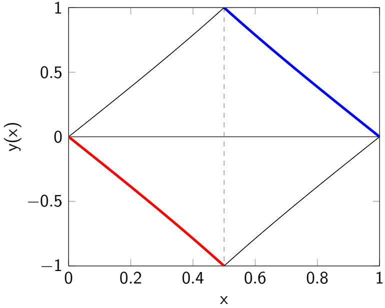
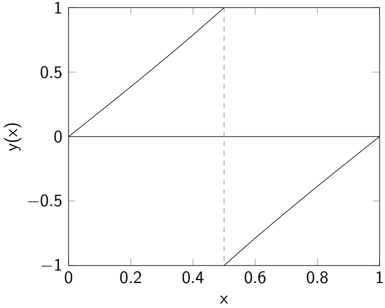


Dirichlet-Neumann with relaxation

Relaxation parameter: $\theta_1, \theta_2 \in (0,1)$.

$$\nu D^{(4)} y_1^k - y_1^k = 0, \qquad \nu D^{(4)} y_2^k - y_2^k = 0,$$
$$y_1^k(\Gamma) = y_\Gamma^{k-1}, \qquad D^{(1)} y_2^k(\Gamma) = D^{(1)} y_1^k(\Gamma),$$
$$D^{(2)} y_1^k(\Gamma) = \tilde{y}_\Gamma^{k-1}, \qquad D^{(3)} y_2^k(\Gamma) = D^{(3)} y_1^k(\Gamma),$$

$y_\Gamma^k := \theta_1 y_2^{k-1}(\Gamma) + (1-\theta_1) y_\Gamma^{k-1}$, $\tilde{y}_\Gamma^k := \theta_2 D^{(2)} y_2^{k-1}(\Gamma) + (1-\theta_2)\tilde{y}_\Gamma^{k-1}$.



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Relaxation parameter: $\theta_1, \theta_2 \in (0,1)$.

$$\nu D^{(4)} y_1^k - y_1^k = 0, \quad \nu D^{(4)} y_2^k - y_2^k = 0,$$
$$y_1^k(\Gamma) = y_\Gamma^{k-1}, \quad D^{(1)} y_2^k(\Gamma) = D^{(1)} y_1^k(\Gamma),$$
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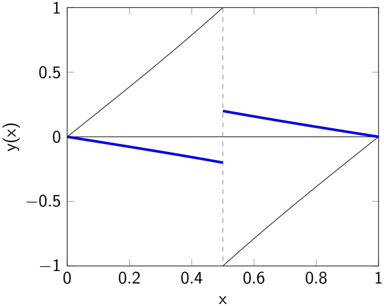

- Solve y_1^k, y_2^k by using $y_1^k(0) = y_2^k(1) = D^{(2)}y_1^k(0) = D^{(2)}y_2^k(1) = 0$.

$$y_1^k(x) = A^k \sinh(\frac{\mu x}{\sqrt{2}}) \cos(\frac{\mu x}{\sqrt{2}}) + B^k \cosh(\frac{\mu x}{\sqrt{2}}) \sin(\frac{\mu x}{\sqrt{2}}),$$

$$y_2^k(x) = C^k \sinh(\frac{\mu(1-x)}{\sqrt{2}}) \cos(\frac{\mu(1-x)}{\sqrt{2}}) + E^k \cosh(\frac{\mu(1-x)}{\sqrt{2}}) \sin(\frac{\mu(1-x)}{\sqrt{2}}).$$

Convergence Analysis

- Solve y_1^k, y_2^k by using $y_1^k(0) = y_2^k(1) = D^{(2)}y_1^k(0) = D^{(2)}y_2^k(1) = 0$.

$$
\begin{aligned}
y_1^k(x) &= A^k \sinh\left(\frac{\mu x}{\sqrt{2}}\right)\cos\left(\frac{\mu x}{\sqrt{2}}\right) + B^k \cosh\left(\frac{\mu x}{\sqrt{2}}\right)\sin\left(\frac{\mu x}{\sqrt{2}}\right),\\
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&\qquad + E^k \cosh\left(\frac{\mu(1-x)}{\sqrt{2}}\right)\sin\left(\frac{\mu(1-x)}{\sqrt{2}}\right).
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$$

- Evaluate A^k, B^k, C^k, E^k by using the transmission condition at Γ.

Convergence Analysis

- Solve y_1^k, y_2^k by using $y_1^k(0) = y_2^k(1) = D^{(2)} y_1^k(0) = D^{(2)} y_2^k(1) = 0$.

$$
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y_1^k(x) &= A^k \sinh(\frac{\mu x}{\sqrt{2}}) \cos(\frac{\mu x}{\sqrt{2}}) + B^k \cosh(\frac{\mu x}{\sqrt{2}}) \sin(\frac{\mu x}{\sqrt{2}}), \\
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&\quad + E^k \cosh(\frac{\mu(1-x)}{\sqrt{2}}) \sin(\frac{\mu(1-x)}{\sqrt{2}}).
\end{aligned}
$$

- Evaluate A^k, B^k, C^k, E^k by using the transmission condition at Γ.
- Different behaviours according to the interface Γ.

Neumann-Neumann (Bourgat, Glowinski, Tallec, Vidrascu 1989)

Primal problem:

$$\begin{aligned} -\Delta y_j^k &= \nu^{-1}\lambda_j^k, && \text{in } \Omega_j & -\Delta \psi_j^k &= 0, && \text{in } \Omega_j \\ y_j^k &= 0, && \text{on } \partial\Omega_j/\ \Gamma & \psi_j^k &= 0, && \text{on } \partial\Omega_j/\ \Gamma \\ y_j^k &= y_\Gamma^{k-1}, && \text{on } \Gamma & \partial_{n_j}\psi_j^k &= \partial_{n_1} y_1^k + \partial_{n_2} y_2^k, && \text{on } \Gamma \end{aligned}$$

with $y_\Gamma^k := y_\Gamma^{k-1} - \theta_1 \left(\psi_1^k|_\Gamma + \psi_2^k|_\Gamma\right)$.

Adjoint problem:

$$\begin{aligned} -\Delta \lambda_j^k &= y_j^k - \hat{y}, && \text{in } \Omega_j & -\Delta \phi_j^k &= 0, && \text{in } \Omega_j \\ \lambda_j^k &= 0, && \text{on } \partial\Omega_j/\ \Gamma & \phi_j^k &= 0, && \text{on } \partial\Omega_j/\ \Gamma \\ \lambda_j^k &= \lambda_\Gamma^{k-1}, && \text{on } \Gamma & \partial_{n_j}\phi_j^k &= \partial_{n_1} \lambda_1^k + \partial_{n_2} \lambda_2^k, && \text{on } \Gamma \end{aligned}$$

with $\lambda_\Gamma^k := \lambda_\Gamma^{k-1} - \theta_2 \left(\phi_1^k|_\Gamma + \phi_2^k|_\Gamma\right)$.

Optimality system

$$\mathcal{L}(y,\lambda,u) = \frac{1}{2}\int_\Omega |y(x)-\hat{y}(x)|^2\,\mathrm{d}x + \frac{\nu}{2}\|u\|^2_{U_{\mathrm{ad}}} + \langle \lambda, -\Delta y - u\rangle$$

- Primal problem:

$$\partial_\lambda L(y,\lambda,u) = 0, \quad \Rightarrow \quad \begin{aligned} -\Delta y &= u \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

- Adjoint problem:

$$\partial_y L(y,\lambda,u) = 0, \quad \Rightarrow \quad \begin{aligned} -\Delta \lambda &= y - \hat{y} \quad \text{in } \Omega, \\ \lambda &= 0 \quad\quad\;\; \text{on } \partial\Omega. \end{aligned}$$

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- Optimality condition:

$$\partial_u L(y,\lambda,u) = 0,$$

with $U_{\mathrm{ad}} := H^{-1}(\Omega)$.

Energy norm

- A linear operator $\mathcal{H} : H^{-1}(\Omega) \to H_0^1(\Omega) \subset L^2(\Omega)$ such that $\mathcal{H}u$ is the unique solution of the variational problem related to (1)

$$\int_\Omega \nabla \mathcal{H}u(x) \cdot \nabla v(x) \, \mathrm{d}x = \langle u, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

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- The norm in $H^{-1}(\Omega)$ which is equivalent to the energy norm

$$\|u\|_{H^{-1}(\Omega)}^2 := \langle u, \mathcal{H}u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \|\nabla y\|_{L^2(Q)}^2.$$

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- The norm in $H^{-1}(\Omega)$ which is equivalent to the energy norm

$$\|u\|^2_{H^{-1}(\Omega)} := \langle u, \mathcal{H}u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \|\nabla y\|^2_{L^2(Q)}.$$

- $\mathcal{H}$ is self-adjoint.

Energy norm

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- The norm in $H^{-1}(\Omega)$ which is equivalent to the energy norm

$$\|u\|_{H^{-1}(\Omega)}^2 := \langle u, \mathcal{H}u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \|\nabla y\|_{L^2(Q)}^2.$$

- $\mathcal{H}$ is self-adjoint.
- Identity: $y = \mathcal{H}u$.

Optimality system

$$\mathcal{L}(y,\lambda,u) = \frac{1}{2}\int_\Omega |y(x)-\hat{y}(x)|^2 \,\mathrm{d}x + \frac{\nu}{2}\langle \mathcal{H}u, u\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} + \langle \lambda, -\Delta y - u\rangle.$$

- Primal problem:

$$\partial_\lambda L(y,\lambda,u) = 0, \quad \Rightarrow \quad \begin{aligned} -\Delta y &= u \quad \text{in } \Omega,\\ y &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

- Adjoint problem:

$$\partial_y L(y,\lambda,u) = 0, \quad \Rightarrow \quad \begin{aligned} -\Delta \lambda &= y-\hat{y} \quad \text{in } \Omega,\\ \lambda &= 0 \qquad\quad \text{on } \partial\Omega. \end{aligned}$$

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$$\partial_u L(y,\lambda,u) = 0 \quad \Rightarrow \quad -\lambda + \nu\mathcal{H}u = 0.$$

Optimality system

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$$-\lambda + \nu\mathcal{H}u = 0.$$

★ First-order optimality system:

$$y = \mathcal{H}u \Leftarrow \begin{aligned} \cancel{-\Delta y = u \text{ in } \Omega,} \qquad & -\Delta\lambda = y - \hat{y} \quad \text{in } \Omega, \\ \cancel{y = 0 \text{ on } \partial\Omega,} \qquad & \lambda = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

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\end{aligned}
$$

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$$
y = \mathcal{H}u \Leftarrow
\begin{aligned}
&\cancel{-\Delta y = u \text{ in } \Omega,} & -\Delta\lambda &= y - \hat{y} \quad \text{in } \Omega,\\
&\cancel{y = 0 \text{ on } \partial\Omega,} & \lambda &= 0 \qquad\quad \text{on } \partial\Omega,
\end{aligned}
$$

$$-\lambda + \nu\mathcal{H}u = 0.$$

★ First-order optimality system:

$$
\begin{aligned}
-\nu\Delta y + y &= \hat{y} \text{ in } \Omega,\\
y &= 0 \text{ on } \partial\Omega.
\end{aligned}
$$

Error Analysis (DN)

Example: $\Omega_1 = (0, \Gamma), \Omega_2 = (\Gamma, 1)$ with Γ the interface.

- Error equation for $e_j^k := y - y_j^k$

$$\nu \ddot{e}_1^k - e_1^k = 0, \quad e_1^k(0) = 0, \quad e_1^k(\Gamma) = e_2^{k-1}(\Gamma),$$
$$\nu \ddot{e}_2^k - e_2^k = 0, \quad e_2^k(1) = 0, \quad \dot{e}_2^k(\Gamma) = \dot{e}_1^k(\Gamma).$$

Error Analysis (DN)

Example: $\Omega_1 = (0, \Gamma), \Omega_2 = (\Gamma, 1)$ with Γ the interface.

- Error equation for $e_j^k := y - y_j^k$

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$$e_1^k(x) = A^k \sinh(\sqrt{\nu^{-1}} x), \quad e_2^k(x) = B^k \sinh\left(\sqrt{\nu^{-1}}(1 - x)\right) e^{-\sqrt{\nu^{-1}}}.$$

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- In particular:

$$\rho_{\text{DN}}\big|_{\Gamma = \frac{1}{2}} = \tanh\left(\sqrt{\nu^{-1}}\frac{1}{2}\right) \coth\left(\sqrt{\nu^{-1}}\frac{1}{2}\right) = 1.$$

Error Analysis (DNR)

- Error equation for $e_j^k := y - y_j^k$

$$\nu \ddot{e}_1^k - e_1^k = 0, \quad e_1^k(0) = 0, \quad e_1^k(\Gamma) = e_\Gamma^{k-1},$$
$$\nu \ddot{e}_2^k - e_2^k = 0, \quad e_2^k(1) = 0, \quad \dot{e}_2^k(\Gamma) = \dot{e}_1^k(\Gamma),$$

with $e_\Gamma^k := (1-\theta)e_\Gamma^{k-1} + \theta e_2^k(\Gamma)$, $\theta \in (0,1)$.

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- Convergence factor:

$$\rho_{\text{DNR}} := \left| 1 - \theta \left[1 + \tanh\left(\sqrt{\nu^{-1}}(1-\Gamma)\right) \coth\left(\sqrt{\nu^{-1}}\Gamma\right) \right] \right|.$$

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Error Analysis (NN)

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$$\nu \ddot{\psi}_j^k - \psi_j^k = 0, \quad \psi_1^k(0) = 0, \quad \psi_2^k(1) = 0, \quad \partial_{n_j}\psi_j^k = \partial_{n_1} e_1^k + \partial_{n_2} e_2^k,$$

with $e_\Gamma^k := e_\Gamma^{k-1} - \theta\left(\psi_1^k(\Gamma) + \psi_2^k(\Gamma)\right)$, $\theta \in (0,1)$.

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$$\rho_{\mathrm{NN}} := \Big| 1 - \theta\Big(\tanh(\sqrt{\nu^{-1}}\Gamma) + \tanh(\sqrt{\nu^{-1}}(1-\Gamma))\Big)\Big(\coth(\sqrt{\nu^{-1}}\Gamma) + \coth(\sqrt{\nu^{-1}}(1-\Gamma))\Big)\Big|.$$

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- In particular:

$$\rho_{\text{NN}}\big|_{\Gamma=\frac{1}{2}} = 1 - 4\theta.$$

Error Analysis 2D (DN)

- Domain: $\Omega_1 = (a, 0) \times \mathbb{R}, \Omega_2 = (0, b) \times \mathbb{R}$ with $x_1 = 0$ the interface and $a < 0 < b$.

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$$
\begin{aligned}
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&\nu \partial_{xx} \hat{e}_2^n - (\nu k^2 + 1)\hat{e}_2^n = 0, \quad \hat{e}_2^n(b, k) = 0, \quad \partial_x e_2^n(0, k) = \partial_x e_1^n(0, k).
\end{aligned}
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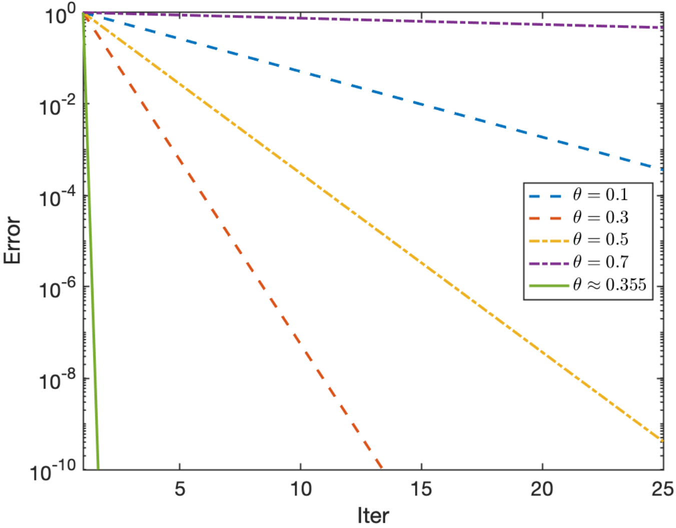
- Parabolic control with semi-discretization:

$$\begin{aligned} &\nu\ddot{e}_1^k - (\nu d_i^2 + 1)e_1^k = 0, \quad e_1^k(0) = 0, \quad e_1^k(\Gamma) = e_\Gamma^{k-1}, \\ &\nu\ddot{e}_2^k - (\nu d_i^2 + 1)e_2^k = 0, \quad \dot{e}_2^k(T) + d_i e_2^k(T) = 0, \quad \dot{e}_2^k(\Gamma) = \dot{e}_1^k(\Gamma). \end{aligned}$$

Convergence tests (DN)



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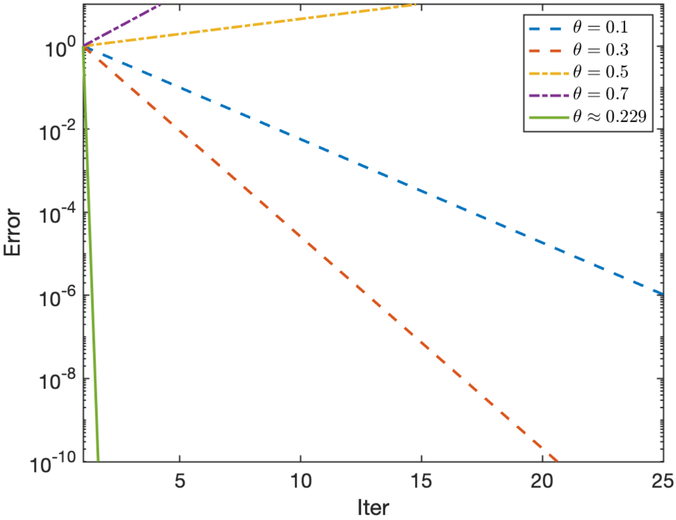

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  Optimal control for Poisson Equation under L^2-regularization:

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- Multi-grid Method for elliptic control (LFA)

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- Multi-grid Method for elliptic control (LFA)

  Poisson equation

$$\rho(\alpha,\omega) = 1 - \alpha(1 - \cos(\omega\Delta x)).$$

  Optimal control for Poisson equation under H^{-2}-regularization:

$$\rho(\alpha,\omega) = 1 - \alpha(1 - \frac{2\cos(\omega\Delta x)}{2+\Delta x^2}).$$

Thanks for your attention !