Dirichlet-Neumann and Neumann-Neumann Methods for Parabolic Optimal Control Problems 2.0

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Joint work with Martin J. Gander

Model problem

For $\hat{y} \in L^2(Q)$, $\gamma \geq 0$, $\nu > 0$ and $\Omega \subset \mathbb{R}^n$, minimize the cost functional

$$J(y,u) := \frac{1}{2}\|y - \hat{y}\|^2_{L^2(Q)} + \frac{\gamma}{2}\|y(T) - \hat{y}(T)\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)},$$

subject to

$$\begin{aligned} \partial_t y - \Delta_x y &= u && \text{in } Q := (0,T) \times \Omega, \\ y &= 0 && \text{on } \Sigma := (0,T) \times \partial\Omega, \\ y &= y_0 && \text{on } \Sigma_0 := \{0\} \times \Omega. \end{aligned}$$

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First-order optimality system (forward-backward):

$$\begin{aligned} \partial_t y - \Delta_x y &= u && \text{in } Q, & \partial_t\lambda + \Delta_x\lambda &= y - \hat{y} && \text{in } Q, \\ y &= 0 && \text{in } \Sigma, & \lambda &= 0 && \text{in } \Sigma, \\ y &= y_0 && \text{in } \Sigma_0, & \lambda &= -\gamma(y-\hat{y}) && \text{in } \Sigma_T := \{T\}\times\Omega, \end{aligned}$$

$$-\lambda + \nu u = 0 \quad \text{in } Q.$$

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Time variable plays a particular role !

Semi-discretization

Ex: finite difference $-\Delta_x \approx A$

$$\begin{cases} \begin{pmatrix} \dot{Y} \\ \dot{\Lambda} \end{pmatrix} + \begin{pmatrix} A & -\nu^{-1} I \\ -I & -A^T \end{pmatrix} \begin{pmatrix} Y \\ \Lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{Y} \end{pmatrix} \text{ in } (0, T), \\ Y(0) = Y_0, \\ \Lambda(T) + \gamma Y(T) = \gamma \hat{Y}(T), \end{cases}$$

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$A = PDP^{-1}$ and $D = \mathrm{diag}(d_1, \ldots, d_n)$,

$$\begin{cases} \begin{pmatrix} \dot{z}_i \\ \dot{\mu}_i \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_i \\ \mu_i \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{z}_i \end{pmatrix} \text{ in } (0, T), \\ z_i(0) = z_{0,i}, \\ \mu_i(T) + \gamma z_i(T) = \gamma \hat{z}_i(T), \end{cases}$$

with $z = P^{-1} Y$, $\hat{z} = P^{-1} \hat{Y}$ and $\mu = P^{-1} \Lambda$.

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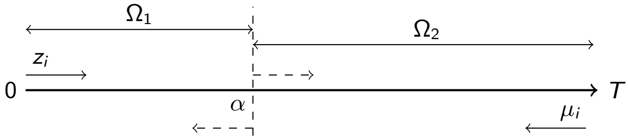
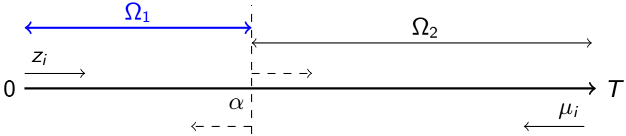


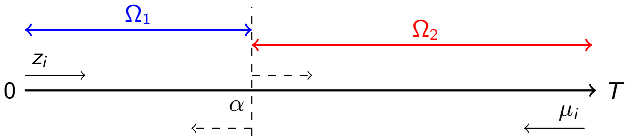


Dirichlet:

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^k \\ \dot{\mu}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^k \\ \mu_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^k(0) = 0, \\ \mu_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}, \end{cases}$$

[1] Bjørstad, Widlund 1986

Dirichlet-Neumann[1]



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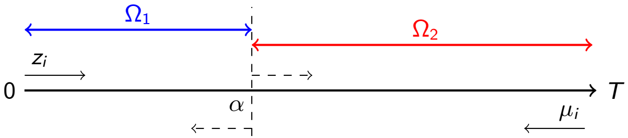

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Update: $f_{\alpha,i}^k := (1-\theta) f_{\alpha,i}^{k-1} + \theta \mu_{2,i}^k(\alpha), \quad \theta \in (0,1).$

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Notation: $\sigma_i := \sqrt{d_i^2 + \nu^{-1}}$, $\omega_i := \nu^{-1}\gamma + d_i$ and $\beta_i := 1 - \gamma d_i$.

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Observations:

- three systems are equivalent,
- same convergence using z or μ,
- not anymore a DN algorithm,
- forward-backward structure less important.

Variants of DN and ND algorithms[2]

Category	Ω_1	Ω_2	type
(z_i, μ_i)	μ_i	$\dot{z}_i$	(DN)
	$\dot{z}_i + d_i z_i$	$\dot{z}_i$	(RN)
	$\dot{\mu}_i$	z_i	(ND)
	$\ddot{z}_i + d_i \dot{z}_i$	z_i	(RD)
z_i	z_i	$\dot{z}_i$	(DN)
	z_i	$\dot{z}_i$	(DN)
	$\dot{z}_i$	z_i	(ND)
	$\dot{z}_i$	z_i	(ND)
μ_i	μ_i	$\dot{\mu}_i$	(DN)
	$\dot{z}_i + d_i z_i$	$\ddot{z}_i + d_i \dot{z}_i$	(RR)
	$\dot{\mu}_i$	μ_i	(ND)
	$\ddot{z}_i + d_i \dot{z}_i$	$\dot{z}_i + d_i z_i$	(RR)

[2] Gander and L. 2023

Natural DN:

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^k \\ \dot{\mu}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^k \\ \mu_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^k(0) = 0, \\ \mu_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}, \end{cases} \quad \begin{cases} \begin{pmatrix} \dot{z}_{2,i}^k \\ \dot{\mu}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^k \\ \mu_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}_{2,i}^k(\alpha) = \dot{z}_{1,i}^k(\alpha), \\ \mu_{2,i}^k(T) + \gamma z_{2,i}^k(T) = 0, \end{cases}$$

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Dirichlet-Neumann (DD27):

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$$f_{\alpha,i}^k := (1-\theta) f_{\alpha,i}^{k-1} + \theta z_{2,i}^k(\alpha).$$

Forward-backward can always be recovered !

Convergence analysis

Convergence factor with analytical form

$$\rho_{\mathrm{DN}_1} := \max_{d_i \in \lambda(A)} \left| 1 - \theta \left(1 - \nu^{-1} \frac{\sigma_i \gamma + \beta_i \tanh(b_i)}{\left(\sigma_i + d_i \tanh(a_i)\right)\left(\omega_i + \sigma_i \tanh(b_i)\right)} \right) \right|,$$

$$\rho_{\mathrm{ND}_1} := \max_{d_i \in \lambda(A)} \left| 1 - \theta \left(1 - \nu^{-1} \frac{\sigma_i \gamma + \beta_i \coth(b_i)}{\left(\sigma_i + d_i \coth(a_i)\right)\left(\omega_i + \sigma_i \coth(b_i)\right)} \right) \right|.$$

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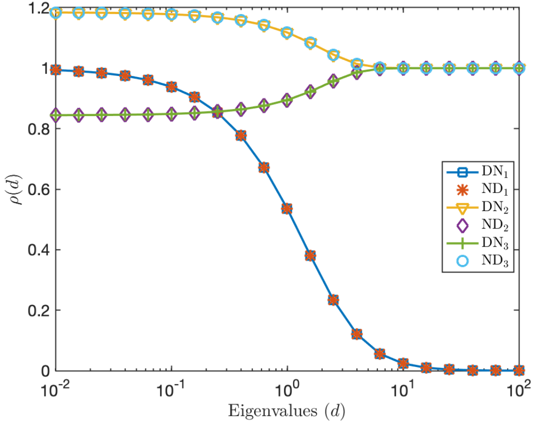
"Optimal" relaxation parameter with equioscillation

$$\theta^*_{\mathrm{DN}_2} = \frac{2}{3 + \coth(\sqrt{\nu^{-1}}\alpha)\frac{\coth\left(\sqrt{\nu^{-1}}(T-\alpha)\right) + \gamma\sqrt{\nu^{-1}}}{1 + \gamma\sqrt{\nu^{-1}}\coth\left(\sqrt{\nu^{-1}}(T-\alpha)\right)}}.$$

$\nu = 0.1$, $\gamma = 0$, $\alpha = \frac{1}{2}$ and $\theta = 1$



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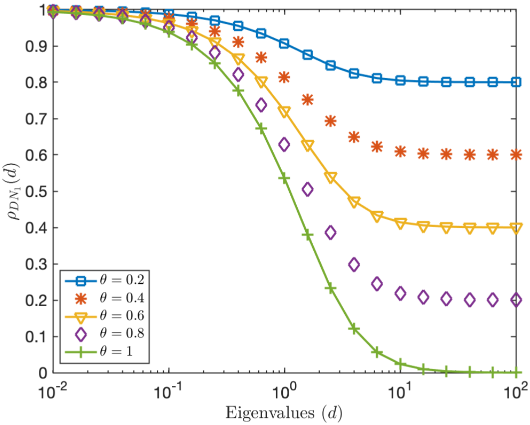
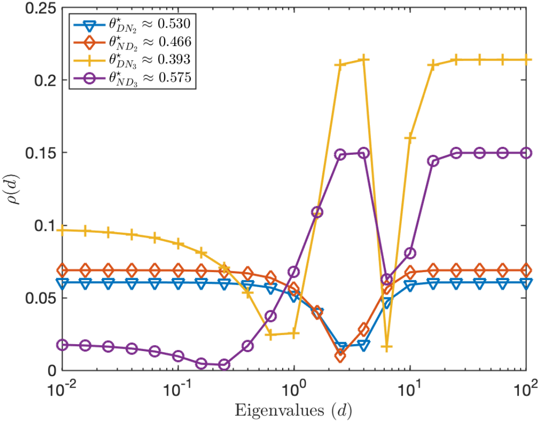


Numerical experiments

$\nu = 0.1$, $\gamma = 10$ and $\alpha = 0.7$



Neumann-Neumann[3]

Dirichlet:

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^k \\ \dot{\mu}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^k \\ \mu_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^k(0) = 0, \\ \mu_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}, \end{cases}$$

$$\begin{cases} \begin{pmatrix} \dot{z}_{2,i}^k \\ \dot{\mu}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^k \\ \mu_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ z_{2,i}^k(\alpha) = g_{\alpha,i}^{k-1}, \\ \mu_{2,i}^k(T) + \gamma z_{2,i}^k(T) = 0, \end{cases}$$

[3] Bourgat, Glowinski, Tallec, Vidrascu 1989

Neumann-Neumann[3]

Neumann:

$$\begin{cases} \begin{pmatrix} \dot{\psi}_{1,i}^k \\ \dot{\phi}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} \psi_{1,i}^k \\ \phi_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ \psi_{1,i}^k(0) = 0, \\ \dot{\phi}_{1,i}^k(\alpha) = \dot{\mu}_{1,i}^k(\alpha) - \dot{\mu}_{2,i}^k(\alpha), \end{cases}$$

$$\begin{cases} \begin{pmatrix} \dot{\psi}_{2,i}^k \\ \dot{\phi}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} \psi_{2,i}^k \\ \phi_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{\psi}_{2,i}^k(\alpha) = \dot{z}_{2,i}^k(\alpha) - \dot{z}_{1,i}^k(\alpha), \\ \phi_{2,i}^k(T) + \gamma\psi_{2,i}^k(T) = 0. \end{cases}$$

[3] Bourgat, Glowinski, Tallec, Vidrascu 1989

Neumann-Neumann[3]

Neumann:

$$\begin{cases} \begin{pmatrix} \dot{\psi}_{1,i}^k \\ \dot{\phi}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} \psi_{1,i}^k \\ \phi_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ \psi_{1,i}^k(0) = 0, \\ \dot{\phi}_{1,i}^k(\alpha) = \dot{\mu}_{1,i}^k(\alpha) - \dot{\mu}_{2,i}^k(\alpha), \end{cases}$$

$$\begin{cases} \begin{pmatrix} \dot{\psi}_{2,i}^k \\ \dot{\phi}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} \psi_{2,i}^k \\ \phi_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{\psi}_{2,i}^k(\alpha) = \dot{z}_{2,i}^k(\alpha) - \dot{z}_{1,i}^k(\alpha), \\ \phi_{2,i}^k(T) + \gamma\psi_{2,i}^k(T) = 0. \end{cases}$$

Update:

$$f_{\alpha,i}^k := f_{\alpha,i}^{k-1} - \theta_1\big(\phi_{1,i}^k(\alpha) + \phi_{2,i}^k(\alpha)\big), \quad g_{\alpha,i}^k := g_{\alpha,i}^{k-1} - \theta_2\big(\psi_{1,i}^k(\alpha) + \psi_{2,i}^k(\alpha)\big), \ \theta_1, \theta_2 > 0.$$

[3] Bourgat, Glowinski, Tallec, Vidrascu 1989

Variants of the NN algorithm[4]

category	step	Ω_1	Ω_2	algorithm type
(z_i, μ_i)	Dirichlet step	μ_i	z_i	(DD)
		$\dot{z}_i + d_i z_i$	z_i	(RD)
	Neumann step	$\dot{\phi}_i$	$\dot{\psi}_i$	(NN)
		$\ddot{\psi}_i + d_i \dot{\psi}_i$	$\dot{\psi}_i$	(RN)
		$\dot{\psi}_i$	$\dot{\psi}_i$	(NN)
		$\dot{\psi}_i$	$\dot{\psi}_i$	(NN)
		$\dot{\phi}_i$	$\dot{\phi}_i$	(NN)
		$\ddot{\psi}_i + d_i \dot{\psi}_i$	$\ddot{\psi}_i + d_i \dot{\psi}_i$	(RR)

[4] Gander and L. 2024

Variants of the NN algorithm[4]

category	step	Ω_1	Ω_2	algorithm type
z_i	Dirichlet step	z_i	z_i	(DD)
		z_i	z_i	(DD)
	Neumann step	$\dot{\psi}_i$	$\dot{\psi}_i$	(NN)
		$\dot{\psi}_i$	$\dot{\psi}_i$	(NN)
		$\dot{\phi}_i$	$\dot{\psi}_i$	(NN)
		$\ddot{\psi}_i + d_i\dot{\psi}_i$	$\dot{\psi}_i$	(RN)
		$\dot{\phi}_i$	$\dot{\phi}_i$	(NN)
		$\ddot{\psi}_i + d_i\dot{\psi}_i$	$\ddot{\psi}_i + d_i\dot{\psi}_i$	(RR)

[4] Gander and L. 2024

Variants of the NN algorithm[4]

category	step	Ω_1	Ω_2	algorithm type
μ_i	Dirichlet step	μ_i	μ_i	(DD)
		$\dot{z}_i + d_i z_i$	$\dot{z}_i + d_i z_i$	(RR)
	Neumann step	$\dot{\phi}_i$	$\dot{\phi}_i$	(NN)
		$\ddot{\psi}_i + d_i\dot{\psi}_i$	$\ddot{\psi}_i + d_i\dot{\psi}_i$	(RR)
		$\dot{\phi}_i$	$\dot{\psi}_i$	(NN)
		$\ddot{\psi}_i + d_i\dot{\psi}_i$	$\dot{\psi}_i$	(RN)
		$\dot{\psi}_i$	$\dot{\psi}_i$	(NN)
		$\dot{\psi}_i$	$\dot{\psi}_i$	(NN)

[4] Gander and L. 2024

Dirichlet:

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^k \\ \dot{\mu}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^k \\ \mu_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^k(0) = 0, \\ \mu_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}, \end{cases}$$

$$\begin{cases} \begin{pmatrix} \dot{z}_{2,i}^k \\ \dot{\mu}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^k \\ \mu_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ z_{2,i}^k(\alpha) = g_{\alpha,i}^{k-1}, \\ \mu_{2,i}^k(T) + \gamma z_{2,i}^k(T) = 0, \end{cases}$$

Adapt update

Neumann:

$$\begin{cases} \begin{pmatrix} \dot{\psi}_{1,i}^k \\ \dot{\phi}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} \psi_{1,i}^k \\ \phi_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ \psi_{1,i}^k(0) = 0, \\ \dot{\psi}_{1,i}^k(\alpha) = \dot{z}_{1,i}^k(\alpha) - \dot{z}_{2,i}^k(\alpha), \end{cases}$$

$$\begin{cases} \begin{pmatrix} \dot{\psi}_{2,i}^k \\ \dot{\phi}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} \psi_{2,i}^k \\ \phi_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{\psi}_{2,i}^k(\alpha) = \dot{z}_{2,i}^k(\alpha) - \dot{z}_{1,i}^k(\alpha), \\ \phi_{2,i}^k(T) + \gamma\psi_{2,i}^k(T) = 0. \end{cases}$$

Update:

$$f_{\alpha,i}^k := f_{\alpha,i}^{k-1} - \theta_1\big(\phi_{1,i}^k(\alpha) + \phi_{2,i}^k(\alpha)\big), \quad g_{\alpha,i}^k := g_{\alpha,i}^{k-1} - \theta_2\big(\psi_{1,i}^k(\alpha) + \psi_{2,i}^k(\alpha)\big).$$

Adapt update

Neumann:

$$\begin{cases} \begin{pmatrix} \dot{\psi}_{1,i}^k \\ \dot{\phi}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} \psi_{1,i}^k \\ \phi_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ \psi_{1,i}^k(0) = 0, \\ \dot{\psi}_{1,i}^k(\alpha) = \dot{z}_{1,i}^k(\alpha) - \dot{z}_{2,i}^k(\alpha), \end{cases}$$

$$\begin{cases} \begin{pmatrix} \dot{\psi}_{2,i}^k \\ \dot{\phi}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} \psi_{2,i}^k \\ \phi_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{\psi}_{2,i}^k(\alpha) = \dot{z}_{2,i}^k(\alpha) - \dot{z}_{1,i}^k(\alpha), \\ \phi_{2,i}^k(T) + \gamma\psi_{2,i}^k(T) = 0. \end{cases}$$

Update:

$$f_{\alpha,i}^k := f_{\alpha,i}^{k-1} - \theta_1\big(\phi_{1,i}^k(\alpha) + \phi_{2,i}^k(\alpha)\big), \quad g_{\alpha,i}^k := g_{\alpha,i}^{k-1} - \theta_2\big(\psi_{1,i}^k(\alpha) + \psi_{2,i}^k(\alpha)\big).$$

Does not converge !

Adapt update

Neumann:

$$\begin{cases} \begin{pmatrix} \dot{\psi}_{1,i}^k \\ \dot{\phi}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} \psi_{1,i}^k \\ \phi_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ \psi_{1,i}^k(0) = 0, \\ \dot{\psi}_{1,i}^k(\alpha) = \dot{z}_{1,i}^k(\alpha) - \dot{z}_{2,i}^k(\alpha), \end{cases}$$

$$\begin{cases} \begin{pmatrix} \dot{\psi}_{2,i}^k \\ \dot{\phi}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} \psi_{2,i}^k \\ \phi_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{\psi}_{2,i}^k(\alpha) = \dot{z}_{2,i}^k(\alpha) - \dot{z}_{1,i}^k(\alpha), \\ \phi_{2,i}^k(T) + \gamma \psi_{2,i}^k(T) = 0. \end{cases}$$

Update:

$$f_{\alpha,i}^k := f_{\alpha,i}^{k-1} - \theta_1\big(\phi_{1,i}^k(\alpha) + \phi_{2,i}^k(\alpha)\big), \quad g_{\alpha,i}^k := g_{\alpha,i}^{k-1} - \theta_2\big(\psi_{1,i}^k(\alpha) + \psi_{2,i}^k(\alpha)\big).$$

Does not converge !

$$f_{\alpha,i}^k = f_{\alpha,i}^{k-1} - \theta_1\big(\dot{\psi}_{1,i}^k(\alpha) + d_i \underline{\psi_{1,i}^k(\alpha)} + \dot{\psi}_{2,i}^k(\alpha) + d_i \underline{\psi_{2,i}^k(\alpha)}\big),$$
$$g_{\alpha,i}^k = g_{\alpha,i}^{k-1} - \theta_2\big(\underline{\psi_{1,i}^k(\alpha)} + \underline{\psi_{2,i}^k(\alpha)}\big).$$

Adapt update

Neumann:

$$\begin{cases} \begin{pmatrix} \dot{\psi}_{1,i}^k \\ \dot{\phi}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} \psi_{1,i}^k \\ \phi_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ \psi_{1,i}^k(0) = 0, \\ \dot{\psi}_{1,i}^k(\alpha) = \dot{z}_{1,i}^k(\alpha) - \dot{z}_{2,i}^k(\alpha), \end{cases}$$

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Update:

$$f_{\alpha,i}^k := f_{\alpha,i}^{k-1} - \theta_1\big(\phi_{1,i}^k(\alpha) + \phi_{2,i}^k(\alpha)\big), \quad g_{\alpha,i}^k := g_{\alpha,i}^{k-1} - \theta_2\big(\psi_{1,i}^k(\alpha) + \psi_{2,i}^k(\alpha)\big).$$

Does not converge !

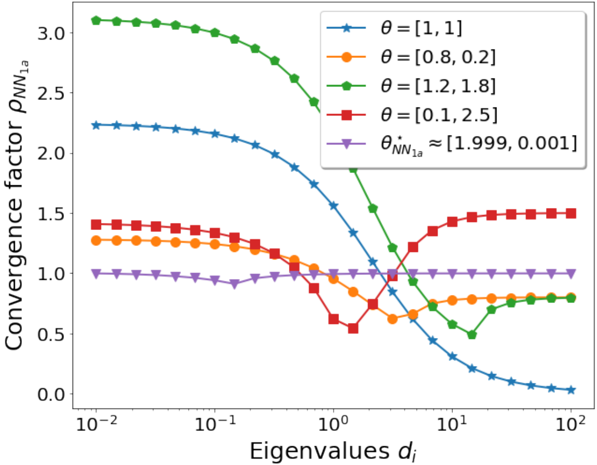
$$f_{\alpha,i}^k = f_{\alpha,i}^{k-1} - \theta_1\big(\dot{\psi}_{1,i}^k(\alpha) + d_i\underline{\psi_{1,i}^k(\alpha)} + \dot{\psi}_{2,i}^k(\alpha) + d_i\underline{\psi_{2,i}^k(\alpha)}\big),$$

$$g_{\alpha,i}^k = g_{\alpha,i}^{k-1} - \theta_2\big(\underline{\psi_{1,i}^k(\alpha)} + \underline{\psi_{2,i}^k(\alpha)}\big).$$

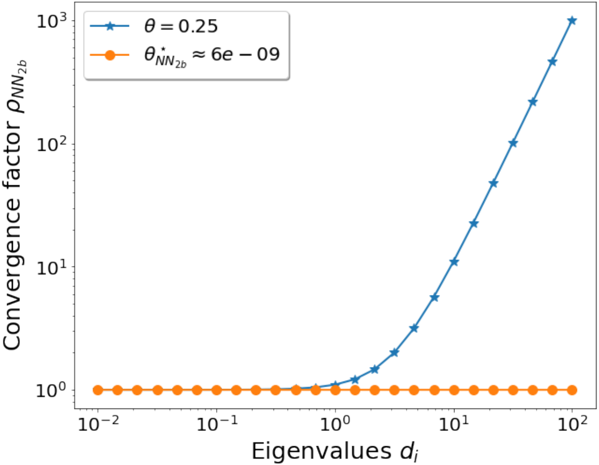
Modified update:

$$f_{\alpha,i}^k \equiv g_{\alpha,i}^k = f_{\alpha,i}^{k-1} - \theta\big(\psi_{1,i}^k(\alpha) + \psi_{2,i}^k(\alpha)\big).$$

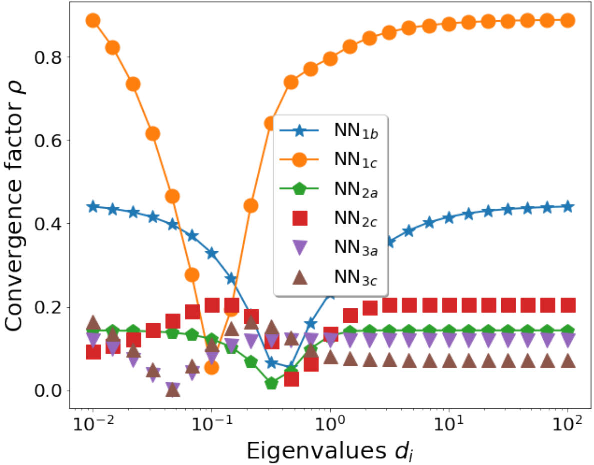
$\nu = 0.1$, $\gamma = 0$ and $\alpha = 0.5$



$\nu = 0.1$, $\gamma = 0$ and $\alpha = 0.5$



$\nu = 10$, $\gamma = 10$ and time domain $(0, 5)$ with $\alpha = 1$



Thanks for your attention !