Shape design combining with a mixing device in an algal raceway pond

Olivier Bernard, Liu-Di LU, Jacques Sainte-Marie, Julien Salomon

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Introduction

- Motivation: High potential on commercial applications, e.g., cosmetics, pharmaceuticals, food complements, wastewater treatment, green energy, etc.
- Raceway ponds

Figure: A typical raceway for cultivating microalgae. Notice the paddle-wheel which mixes the culture suspension. Picture from INRA (ANR Symbiose project) [1].

1D Illustration

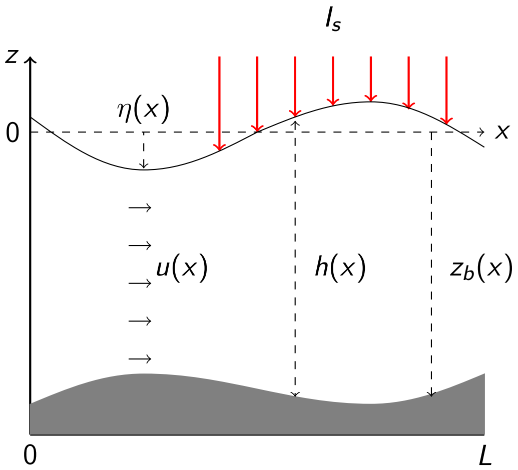

Figure: Representation of 1D raceway.

Saint-Venant Equations

- 1D steady state Saint-Venant equations

$$\partial_x(hu) = 0, \tag{1}$$

$$\partial_x(hu^2 + g\frac{h^2}{2}) = -gh\partial_x z_b. \tag{2}$$

Saint-Venant Equations

- u, z_b as a function of h

$$u = \frac{Q_0}{h}, \tag{1}$$
$$z_b = \frac{M_0}{g} - \frac{Q_0^2}{2gh^2} - h, \tag{2}$$

  $Q_0, M_0 \in \mathbb{R}^+$ are two constants.

- Froude number:

$$Fr := \frac{u}{\sqrt{gh}}$$

  $Fr < 1$: subcritical case (i.e. the flow regime is fluvial)
  $Fr > 1$: supercritical case (i.e. the flow regime is torrential)

- Given a smooth topography z_b, there exists a unique positive smooth solution of h which satisfies the subcritical flow condition [5, Lemma 1].

Lagrangian Trajectories

- Incompressibility of the flow: $\nabla \cdot \underline{\mathbf{u}} = 0$ with $\underline{\mathbf{u}} = (u(x), w(x,z))$

$$\partial_x u + \partial_z w = 0. \tag{3}$$

- Integrating (3) from z_b to z and using the kinematic condition at bottom ($w(x, z_b) = u(x)\partial_x z_b$) gives:

$$w(x,z) = (\frac{M_0}{g} - \frac{3u^2(x)}{2g} - z)u'(x).$$

- The Lagrangian trajectory is characterized by the system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} u(x(t)) \\ w(x(t), z(t)) \end{pmatrix}.$$

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- A time free formulation of the Lagrangian trajectory:

$$z(x) = \color{red}{\eta(x)} + \frac{\color{red}{h(x)}}{\color{blue}{h(0)}}(\color{blue}{z(0)} - \color{blue}{\eta(0)}). \tag{4}$$

Han model [4]

- A: open and ready to harvest a photon,
  B: closed while processing the absorbed photon energy,
  C: inhibited if several photons have been absorbed simultaneously.
-
$$\begin{cases} \dot{A} = -\sigma I A + \frac{B}{\tau}, \\ \dot{B} = \sigma I A - \frac{B}{\tau} + k_r C - k_d \sigma I B, \\ \dot{C} = -k_r C + k_d \sigma I B. \end{cases} \tag{5}$$
- A, B, C are the relative frequencies of the three possible states with $A + B + C = 1$.

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- Using their sum equals to one to eliminate B

$$\left\{\begin{array}{l} \dot{A} = -(\sigma I + \frac{1}{\tau})A + \frac{1-C}{\tau}, \\ \dot{C} = -(k_r + k_d \sigma I)C + k_d \sigma I(1-A), \end{array}\right.$$

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- Using fast-slow approximation, (5) can be reduced to:

$$\dot{C} = -(k_d \tau \frac{(\sigma I)^2}{\tau \sigma I + 1} + k_r) C + k_d \tau \frac{(\sigma I)^2}{\tau \sigma I + 1}.$$

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$$\dot{C} = -(k_d \tau \frac{(\sigma I)^2}{\tau \sigma I + 1} + k_r) C + k_d \tau \frac{(\sigma I)^2}{\tau \sigma I + 1}.$$
- The net growth rate:
$$\mu(C, I) := k \sigma I A - R = k \sigma I \frac{(1 - C)}{\tau \sigma I + 1} - R,$$

Light intensity

The Beer-Lambert law describes how light is attenuated with depth

$$I(x,z) = I_s \exp\Big(-\varepsilon(\eta(x) - z)\Big), \tag{6}$$

where ε is the light extinction defined by:

$$\varepsilon = \frac{1}{h} \ln(\frac{I_s}{I_{z_b}}).$$

Optimization Problem

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$$\bar{\mu}_\infty := \frac{1}{V}\int_0^L \int_{z_b(x)}^{\eta(x)} \mu\big(C(x,z), I(x,z)\big)\mathrm{d}z\mathrm{d}x,$$

$$\bar{\mu}_{N_z} := \frac{1}{VN_z}\sum_{i=1}^{N_z}\int_0^L \mu(C_i, I_i) h\mathrm{d}x.$$

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- Volume of the system

$$V = \int_0^L h(x)\mathrm{d}x. \tag{7}$$

- Parameterize h by a vector $a := [a_1, \cdots, a_N] \in \mathbb{R}^N$.

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- Parameterize h by a vector $a := [a_1, \cdots, a_N] \in \mathbb{R}^N$.
- The computational chain:

$$a \to h \to z_i \to I_i \to C_i \to \bar{\mu}_{N_z}.$$

- Optimization Problem: $\bar{\mu}_{N_z}(a) = \frac{1}{VN_z}\sum_{i=1}^{N_z}\int_0^L \mu(C_i, I_i(a))h(a)\mathrm{d}x$, where C_i satisfy

$$C_i' = \left(-\alpha\left(I_i(a)\right)C_i + \beta\left(I_i(a)\right)\right)\frac{h(a)}{Q_0}.$$

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- Lagrangian

$$\begin{aligned}\mathcal{L}(C_i, a, p_i) = \frac{1}{VN_z}\sum_{i=1}^{N_z}\int_0^L \Big(-\gamma(I_i(a))C_i + \zeta(I_i(a))\Big)h(a)\mathrm{d}x \\ -\sum_{i=1}^{N_z}\int_0^L p_i\Big(C_i' + \frac{\alpha(I_i(a)) - \beta(I_i(a))}{Q_0}h(a)\Big)\mathrm{d}x.\end{aligned}$$

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$$-\sum_{i=1}^{N_z}\int_0^L p_i\Big(C_i' + \frac{\alpha(I_i(a)) - \beta(I_i(a))}{Q_0}h(a)\Big)\mathrm{d}x.$$

- The gradient $\nabla\bar{\mu}_{N_z}(a) = \partial_a\mathcal{L}$ is given by

$$\partial_a\mathcal{L} = \sum_{i=1}^{N_z}\int_0^L \Big(\frac{-\gamma'\left(I_i\right)C_i + \zeta'\left(I_i\right)}{VN_z} + p_i\frac{-\alpha'(I_i)C_i + \beta'(I_i)}{Q_0}\Big)h\partial_a I_i\mathrm{d}x$$
$$+\sum_{i=1}^{N_z}\int_0^L \Big(\frac{-\gamma\left(I_i\right)C_i + \zeta\left(I_i\right)}{VN_z} + p_i\frac{-\alpha(I_i)C_i + \beta(I_i)}{Q_0}\Big)\partial_a h\mathrm{d}x.$$

Numerical settings

Parameterization of h: Truncated Fourier

$$h(x) = a_0 + \sum_{n=1}^{N} a_n \sin(2n\pi \frac{x}{L}). \tag{8}$$

Parameter to be optimized: Fourier coefficients $a := [a_1, \ldots, a_N]$. We use this parameterization based on the following reasons :

- We consider a hydrodynamic regime where the solutions of the shallow water equations are smooth and hence the water depth can be approximated by (8).
- One has naturally $h(0) = h(L)$ under this parameterization, which means that we have accomplished one lap of the raceway pond.
- We assume a constant volume of the system V, which can be achieved by fixing a_0. Indeed, under this parameterization and using (7), one finds $V = a_0 L$.

Convergence

We fix $N = 5$ and take 100 random initial guesses of a. For N_z varying from 1 to 80, we compute the average value of $\bar{\mu}_{N_z}$ for each N_z.

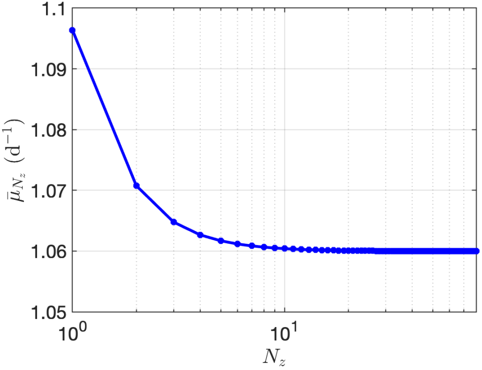
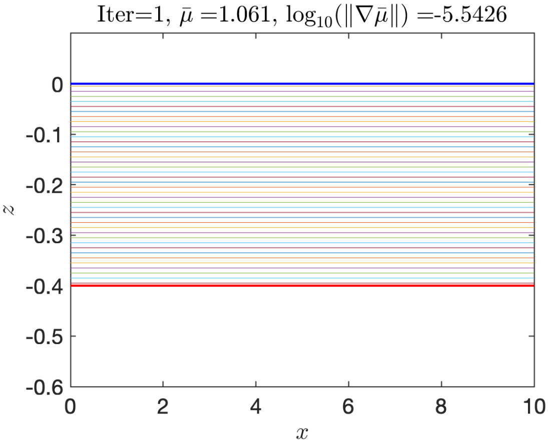


Figure: The value of $\bar{\mu}_{N_z}$ for $N_z = [1, 80]$.

Optimal Topography

We take $N_z = 40$. As an initial guess, we consider the flat topography, meaning that a is set to 0.



Periodic case

Assumption

Photoinhibition state C is periodic meaning that $C_i(L) = C_i(0)$

Consequence

Differentiating $\mathcal{L}$ with respect to $C_i(L)$, we have

$$\partial_{C_i(L)}\mathcal{L} = p_i(L) - p_i(0).$$

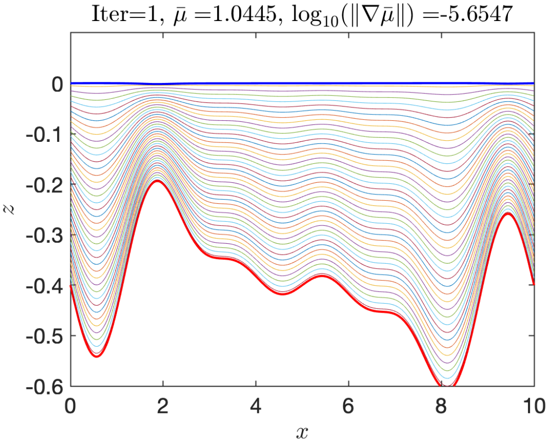
so that equating the above equation to zero gives the periodicity for p_i.

Theorem (Flat topography [2])

Assume the volume of the system V is constant. Then $\nabla \bar{\mu}_{N_z}(0) = 0$.

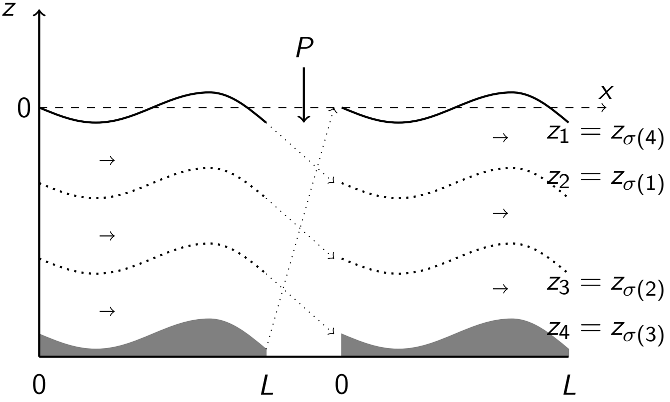
Optimal topography (C periodic)

We keep $N_z = 40$. As an initial guess, we consider a random topography.



Mixing devices

- An ideal rearrangement of trajectories: at each new lap, the algae at depth $z_i(0)$ are entirely transferred into the position $z_j(0)$ when passing through the mixing device.
- We denote by $\mathcal{P}$ the set of permutation matrices of size $N \times N$ and by $\mathfrak{S}_N$ the associated set of permutations of N elements.



Test with a permutation

We keep $N_z = 40$ and choose $\sigma = (1\ N_z)(2\ N_z - 1)\dots$

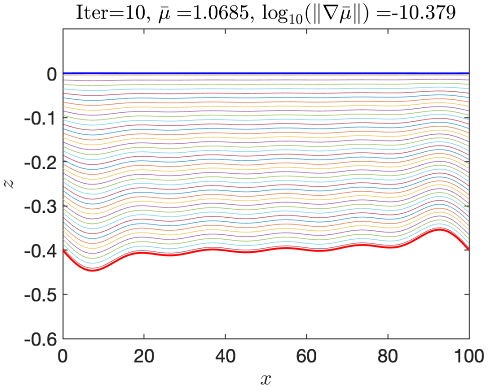


Figure: The optimal topography for two laps.

Test with a permutation

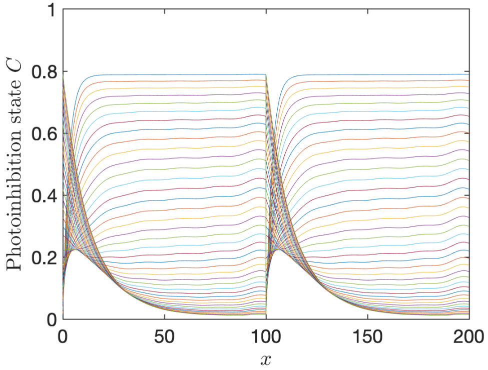


Figure: The evolution of the photo-inhibition state C for two laps.

It has been shown in [3] that if the system is periodic, then the period equals to one.

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- Optimization Problem:

$$\max_{P\in\mathcal{P}} \max_{a\in\mathbb{R}^N} \bar{\mu}^P_{N_z}(a) = \max_{P\in\mathcal{P}} \max_{a\in\mathbb{R}^N} \frac{1}{VN_z} \sum_{i=1}^{N_z} \int_0^L \mu(C_i^P, I_i(a))h(a)\mathrm{d}x,$$

where C_i^P satisfy

$$C_i^{P\prime} = \left(-\alpha\left(I_i(a)\right)C_i^P + \beta\left(I_i(a)\right)\right)\frac{h(a)}{Q_0},$$
$$PC^P(L) = C^P(0).$$

- Lagrangian multiplier

$$p_i^{P\prime} = p_i^P\alpha(I_i(a))\frac{h(a)}{Q_0} - \frac{h(a)}{VN_z}\gamma(I_i(a)),$$
$$p^P(L) = p^P(0)P.$$

Optimal Topography (Constant volume)

We take $N_z = 7$. As an initial guess, we consider the flat topography, meaning that a is set to 0.

$$P^{100}_{\max} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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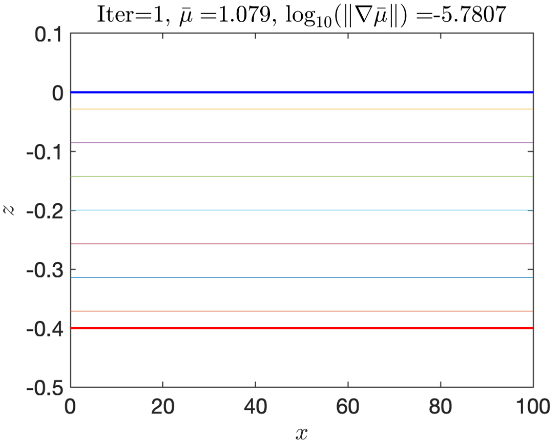

Variable volume

- Volume related parameter a_0 as the average depth of the raceway system:

$$a_0 := \bar{h} = \frac{1}{L}\int_0^L h(x)\mathrm{d}x = \frac{V}{L}. \tag{9}$$

New parameter $\tilde{a} = [a_0, a_1, \ldots, a_N]$.

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New parameter $\tilde{a} = [a_0, a_1, \ldots, a_N]$.
- Optimization Problem:

$$\Pi_{N_z}(\tilde{a}) := \bar{\mu}_{N_z}(\tilde{a})Xh(\tilde{a}) = \frac{Y_{\mathrm{opt}} - \alpha_1 a_0}{VN_z\alpha_0}\sum_{i=1}^{N_z}\int_0^L \mu(C_i^P, I_i(\tilde{a}))h(\tilde{a})\mathrm{d}x$$

where C_i^P satisfy

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- Extra element in gradient: $\nabla\Pi_{N_z}(\tilde{a}) = [\partial_{a_0}\mathcal{L}, \partial_a\mathcal{L}]$.

Optimal Topography (Variable volume)

We keep $N_z = 7$. As an initial guess, we consider the flat topography with $a_0 = 0.4$.

$$P_{\max}^{100} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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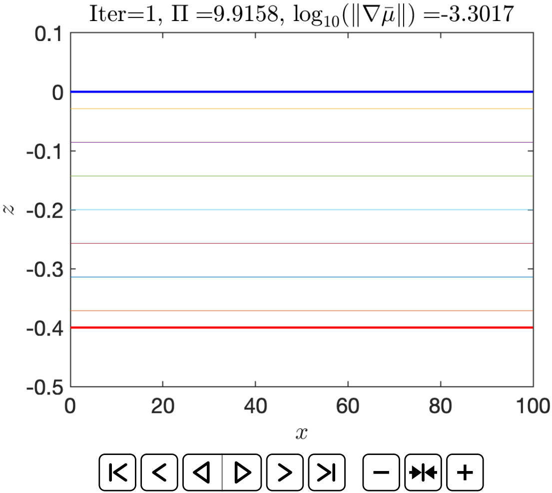

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