# Multigrid method for optimal control problem

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Section of Mathematics
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Lugano, August 25th, 2022

# Motivation

Gander and Neumüller, *Analysis of a new space-time parallel multigrid algorithm for parabolic problems*, SIAM J. SCI. COMPUT., 38(4), A2173 – A2208, 2016

**Model**: heat equation

$$\begin{aligned} \partial_t u(x,t) - \Delta_x u(x,t) &= f(x,t) & (x,t) &\in Q := \Omega \times (0,T), \\ u(x,t) &= 0 & (x,t) &\in \Sigma := \Gamma \times (0,T), \\ u(x,0) &= u_0(x) & (x,t) &\in \Sigma_0 := \Omega \times \{0\}. \end{aligned}$$

**Discretization**: high order discontinuous Galerkin in time and finite element in space.

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Gander and Lunet, *Time Parallel Time Integration*, In preparation, 2022

# Problem

★ **One dimensional case**:

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★ **Discretization**: backward Euler in time and centered finite difference in space

$$\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} = L\mathbf{u}_{n+1} + \mathbf{f}_{n+1},$$

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★ **Two-grid**:

$$\begin{aligned}
\mathbf{u}^{k+\frac{1}{3}} &= S(\mathbf{b}, \mathbf{u}^k, \nu_1), \\
\mathbf{u}^{k+\frac{2}{3}} &= \mathbf{u}^{k+\frac{1}{3}} + PA_c^{-1}R(\mathbf{b} - A\mathbf{u}^{k+\frac{1}{3}}), \\
\mathbf{u}^{k+1} &= S(\mathbf{b}, \mathbf{u}^{k+\frac{2}{3}}, \nu_2),
\end{aligned}$$

with $\nu_1$ number of pre-smoothing step, $\nu_2$ number of post-smoothing step, $P$ prolongation matrix, $R$ restriction matrix and $A_c$ coarse Galerkin matrix.

# Coarsen in space (Hackbusch 1984)

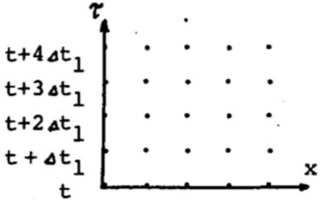


Fig 2.1a: Grid at level l

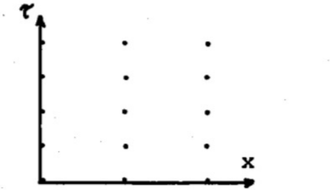


Fig 2.1b: Grid at level l-1

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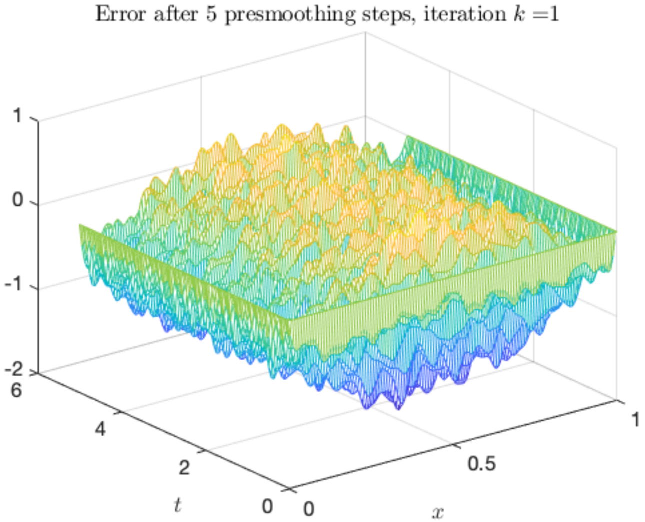

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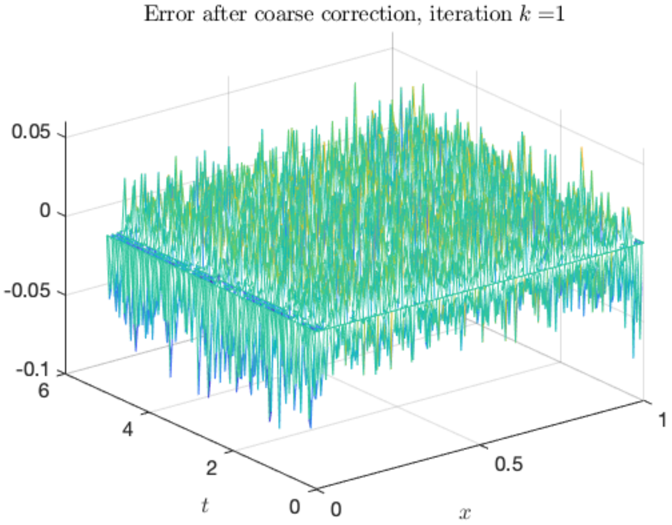

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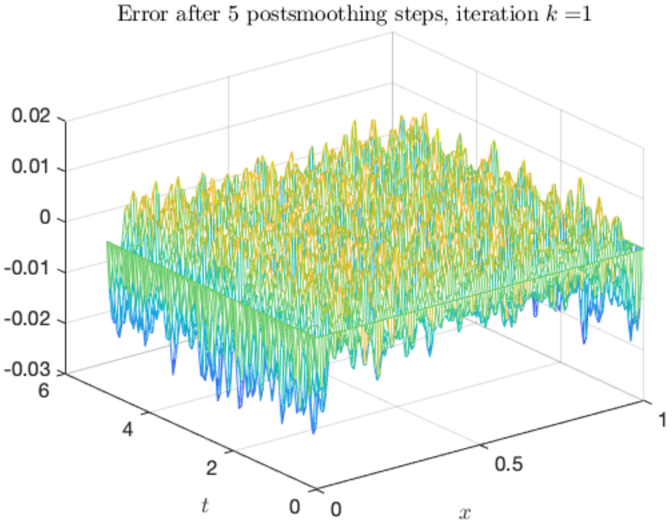

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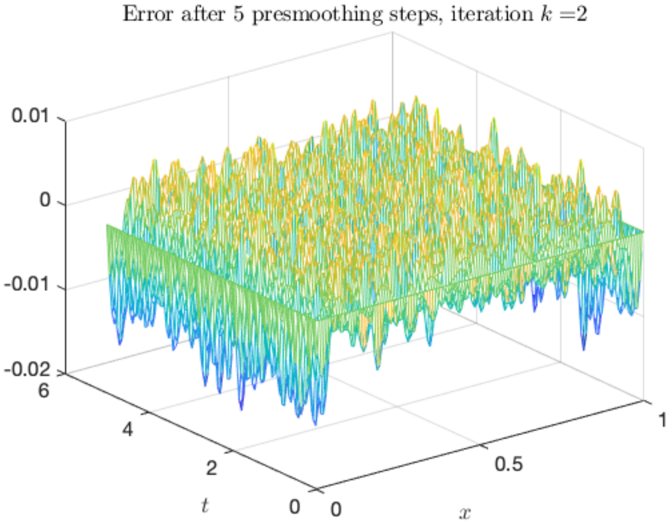

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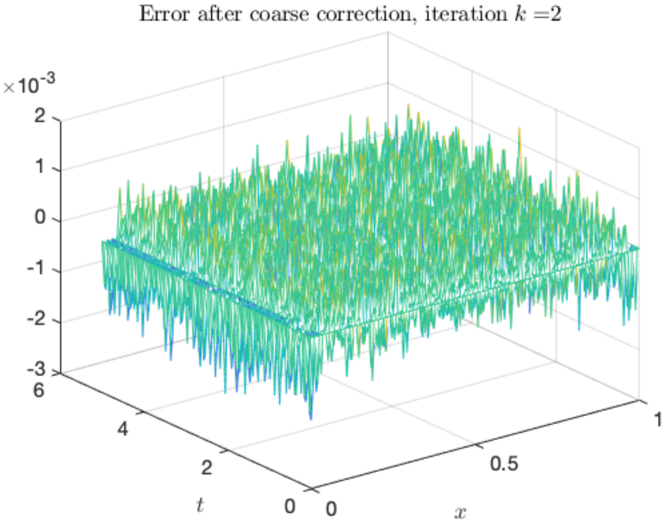

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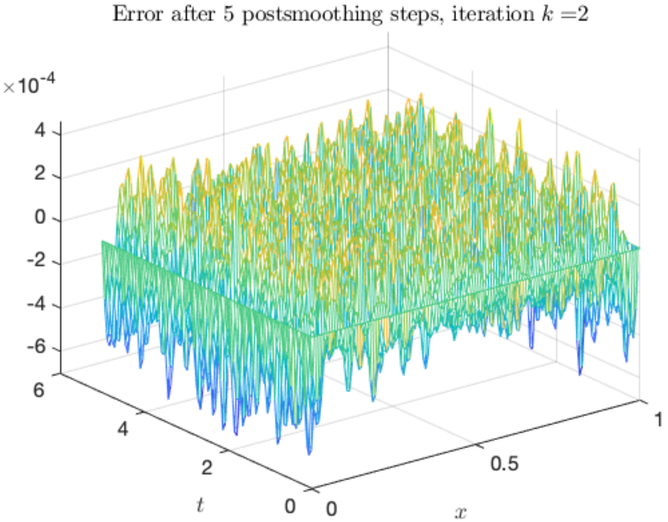

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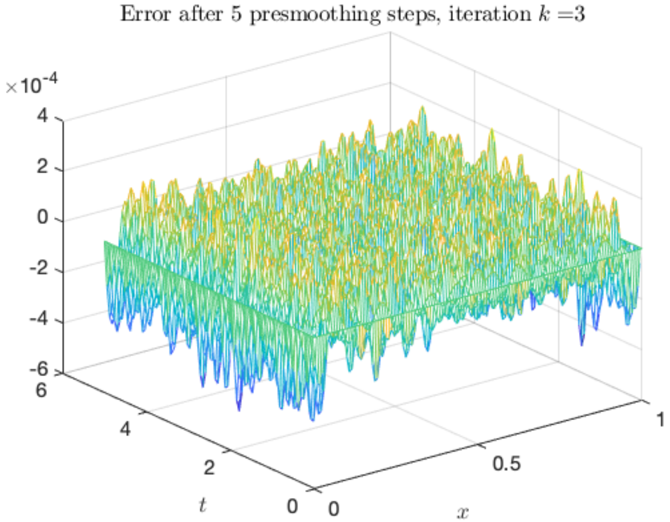

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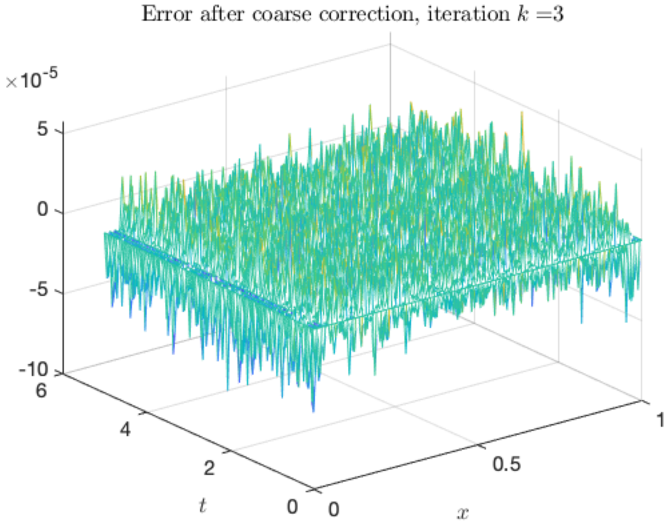

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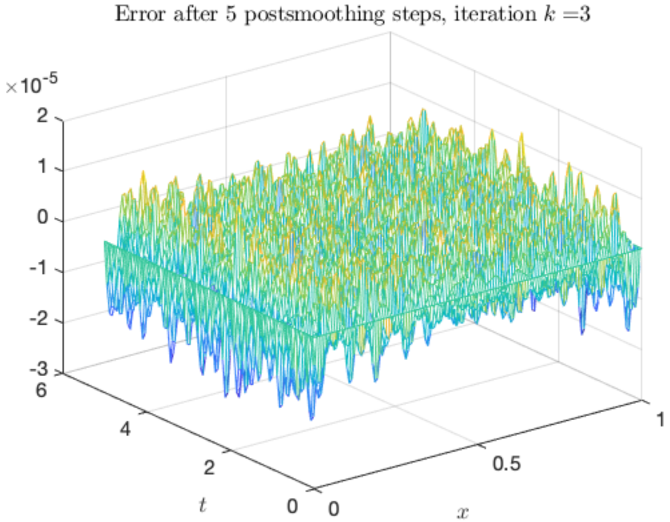

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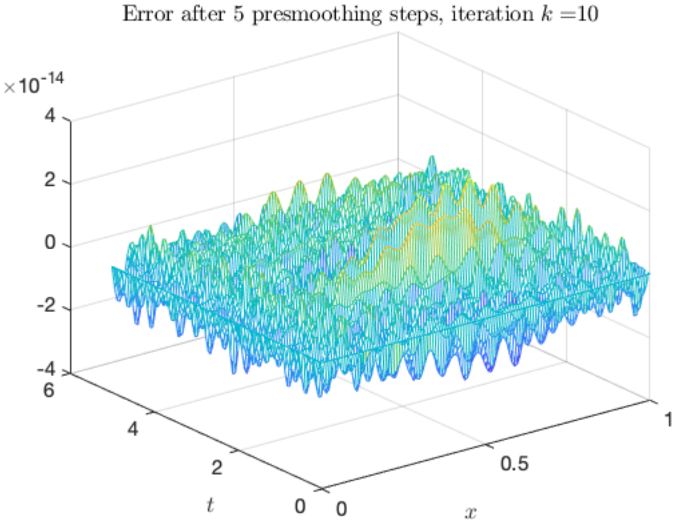

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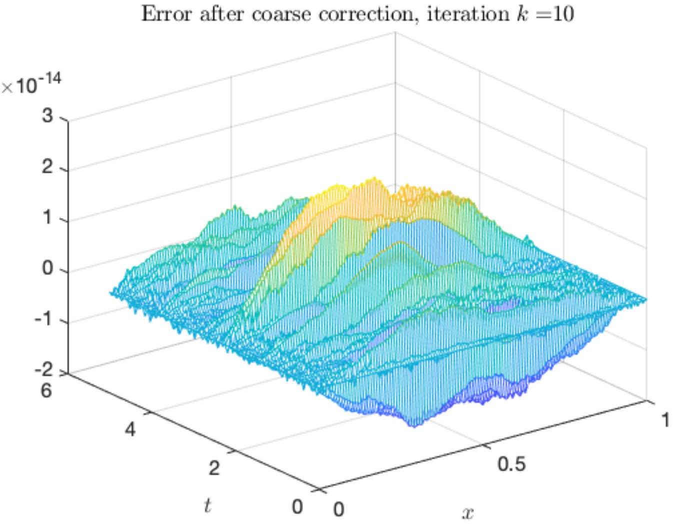

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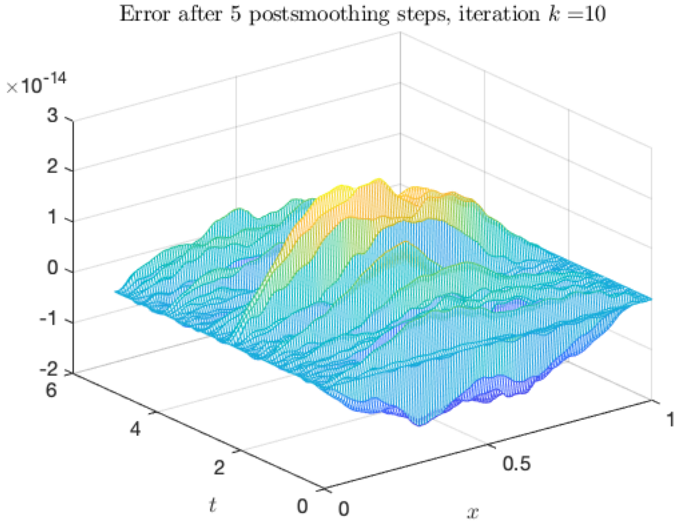

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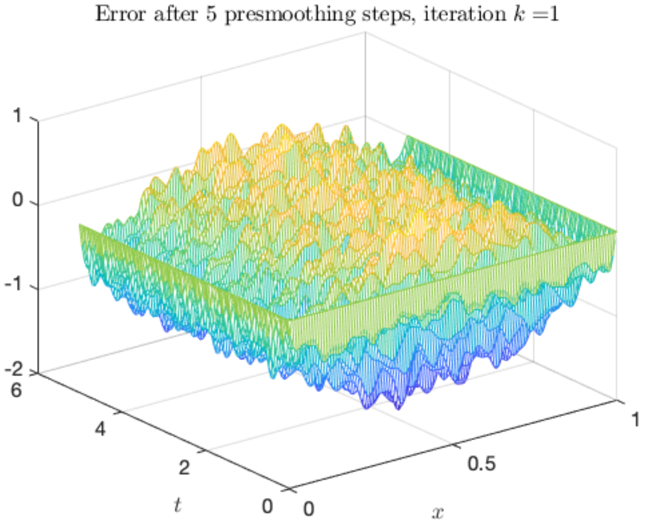

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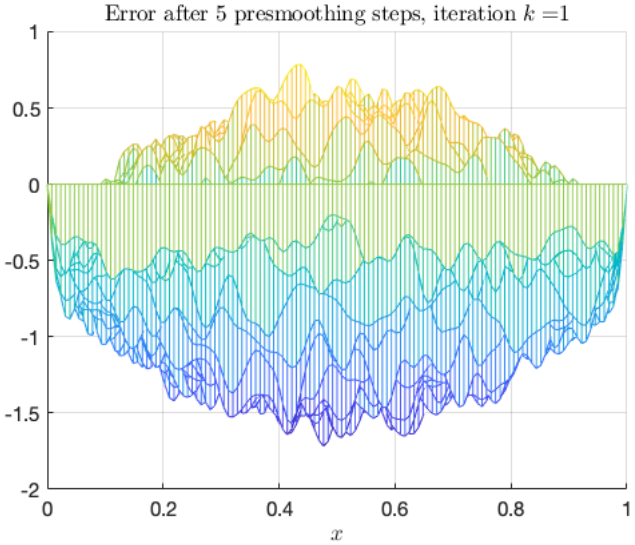

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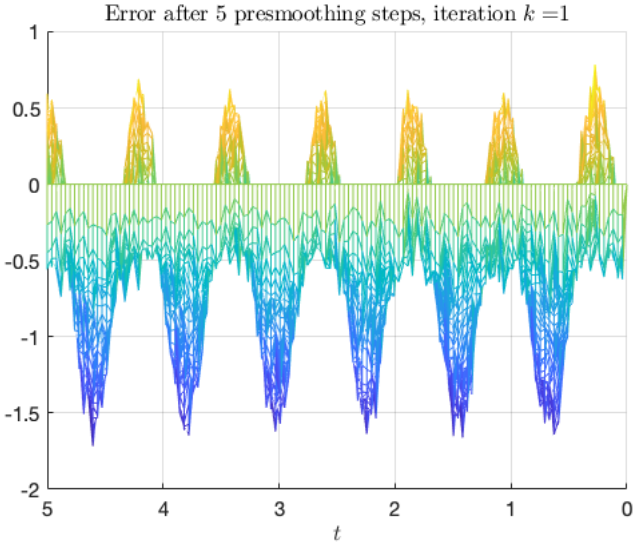

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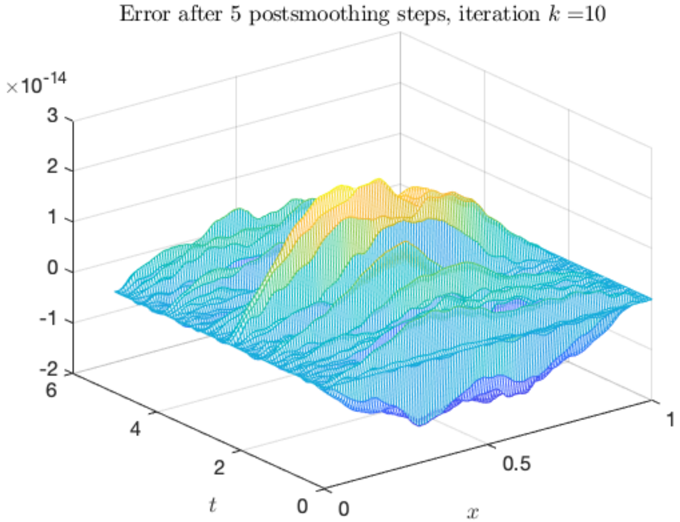

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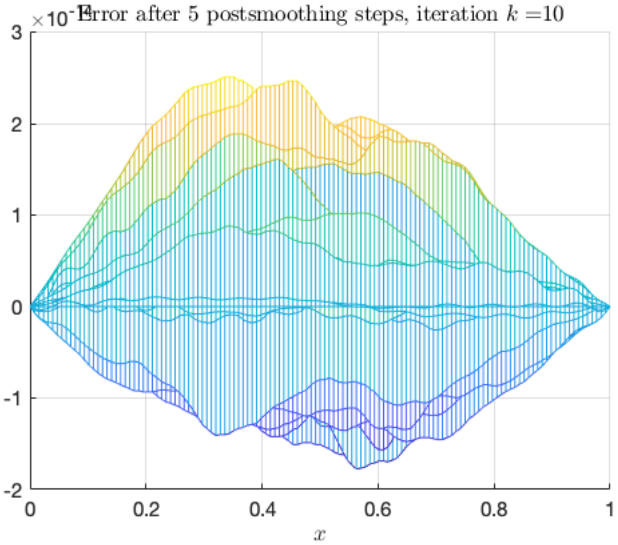

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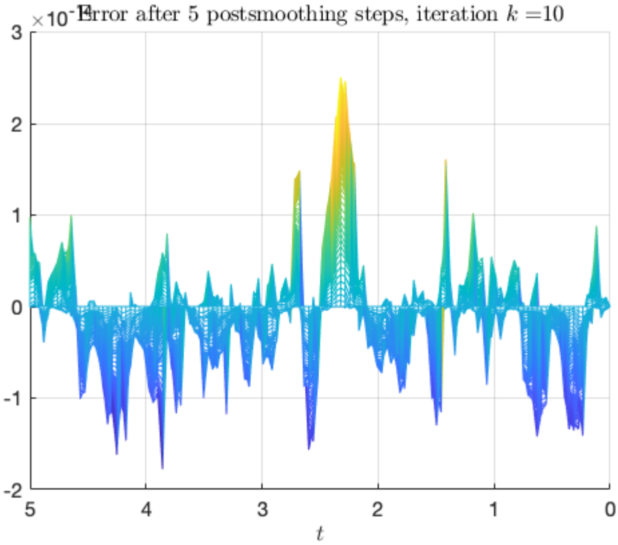

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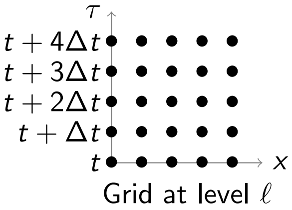


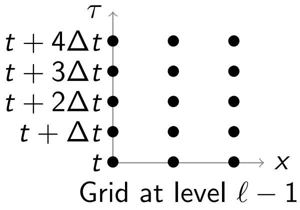


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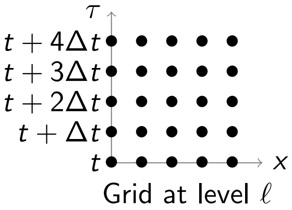


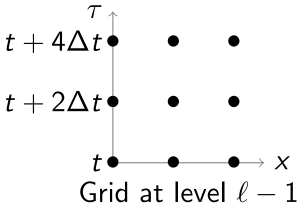

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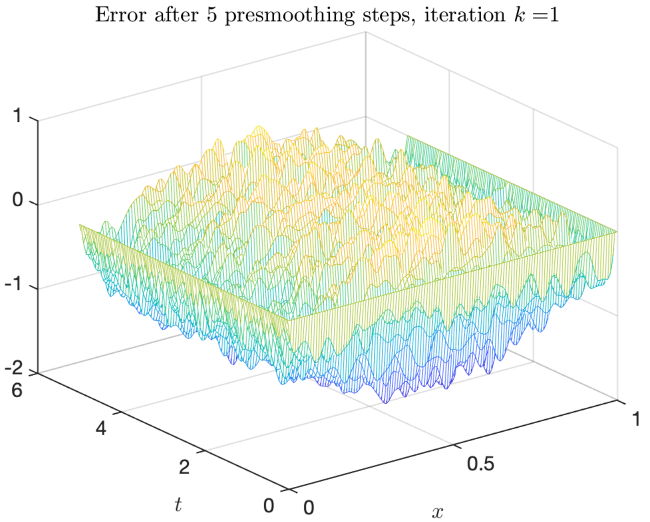

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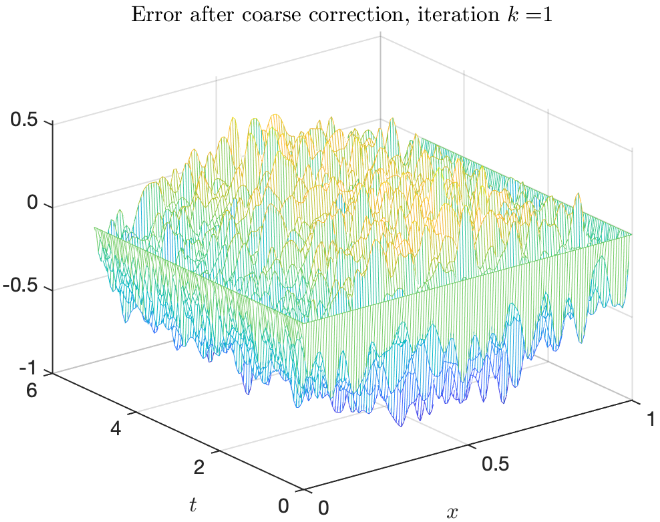

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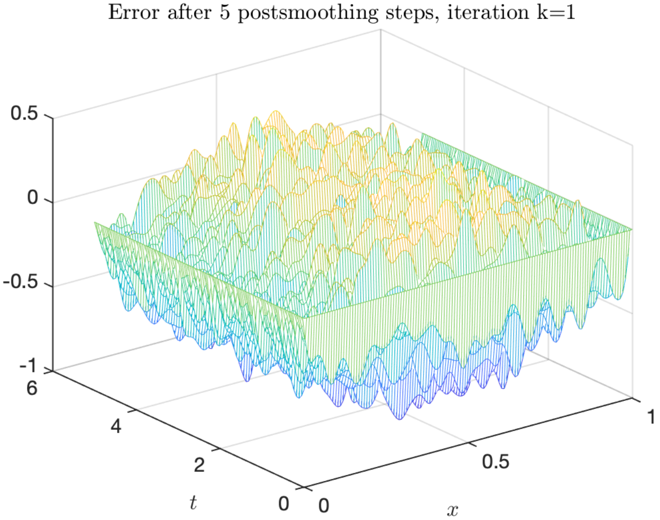

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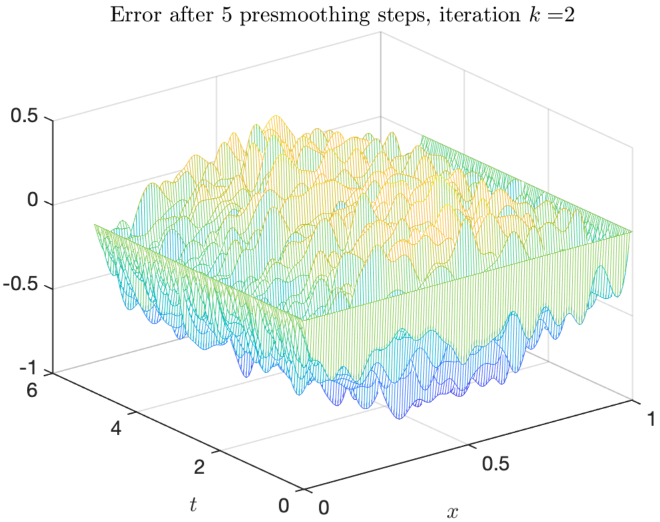

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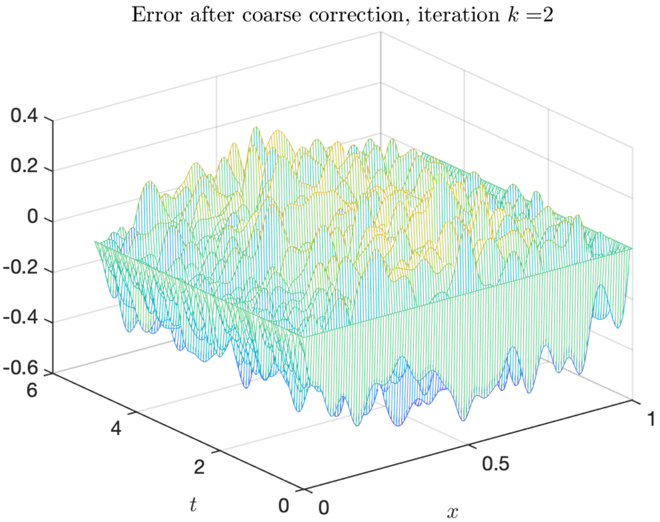

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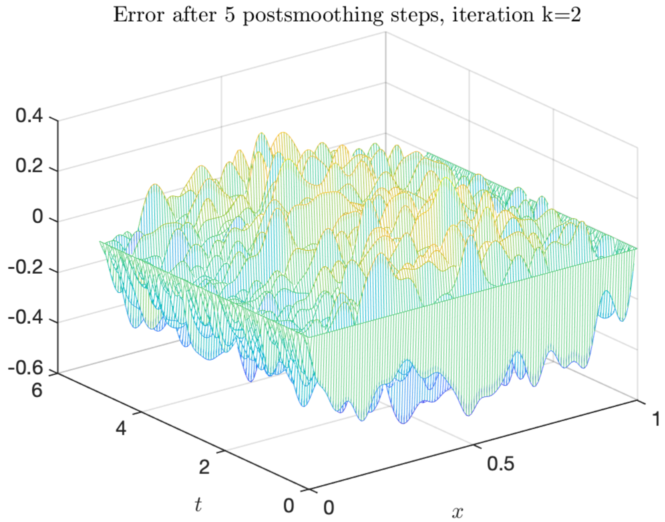

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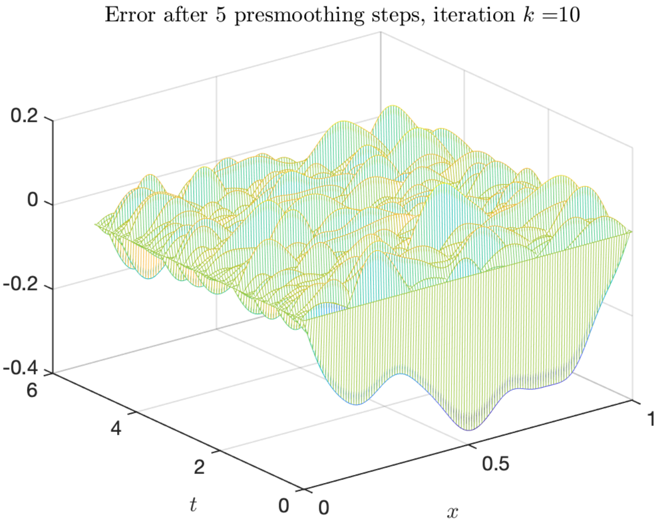

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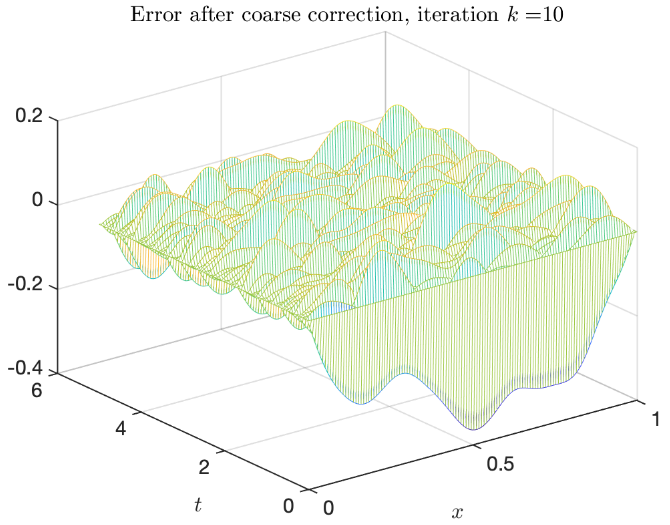

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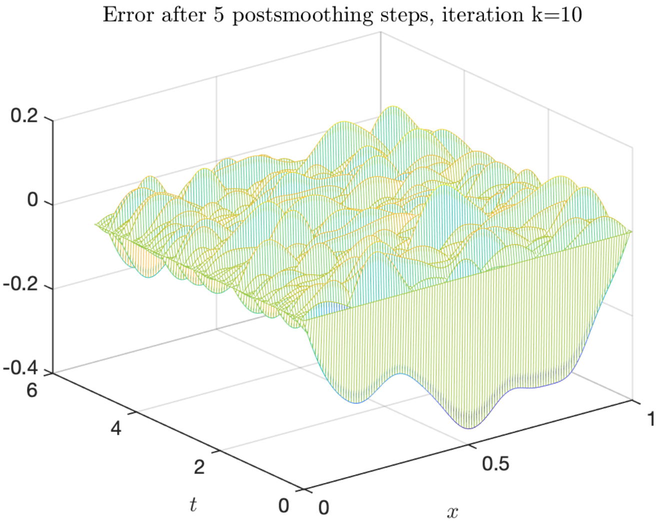

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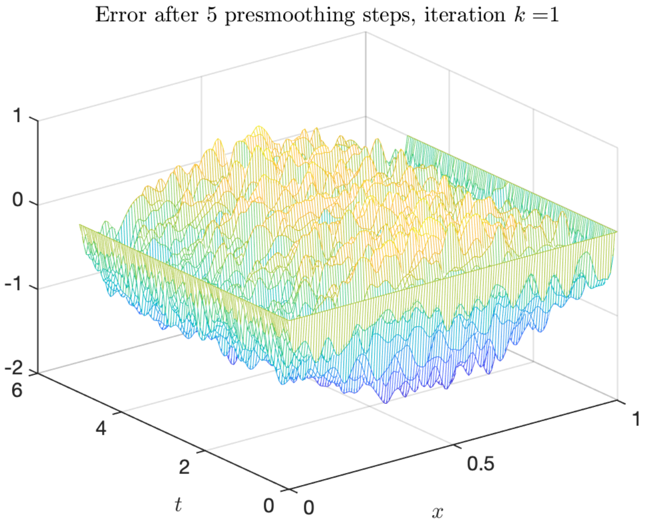

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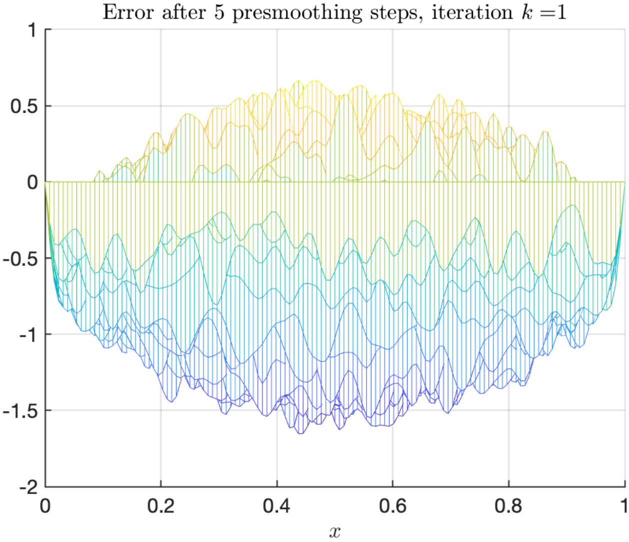

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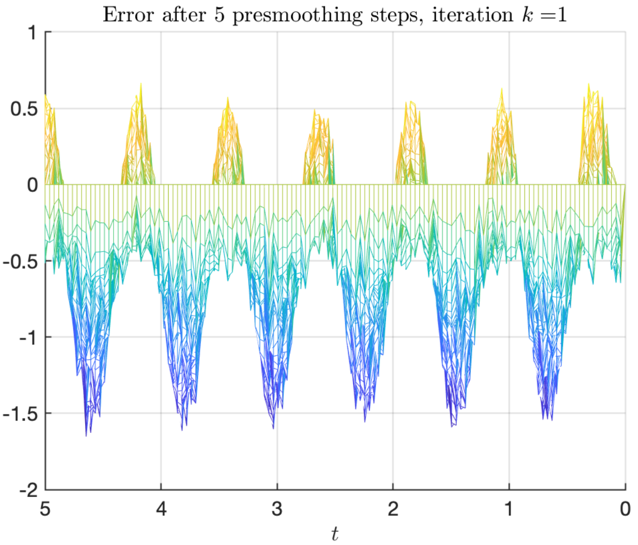

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- Neglect the initial and boundary conditions of the problem, set $\mathbf{b}$ to zero.

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- Convergence factor:

$$\rho(\alpha, \omega, \xi) := 1 - \alpha\left(1 - \frac{2\frac{\Delta t}{\Delta x^2}\cos(\omega \Delta x) + e^{-i\xi\Delta t}}{1 + 2\frac{\Delta t}{\Delta x^2}}\right).$$

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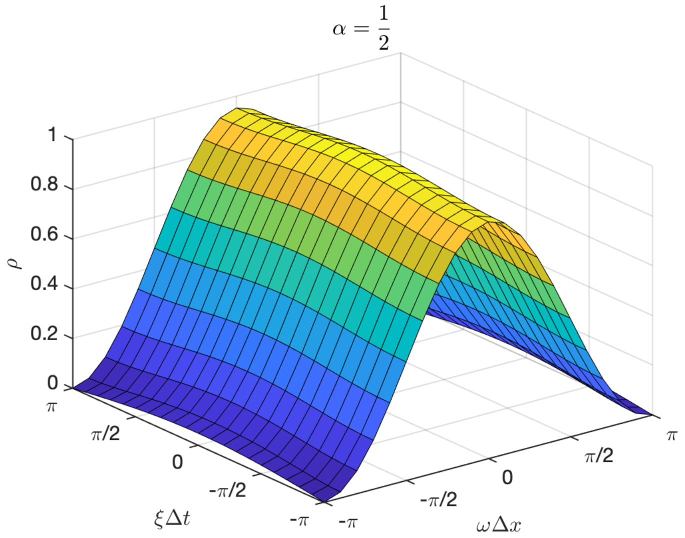

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★ **Smoother**: damped Jacobi

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \alpha D^{-1}(\mathbf{b} - A\mathbf{u}^k) = \mathbf{u}^k - \frac{\alpha}{1 - \lambda \Delta t} A\mathbf{u}^k$$

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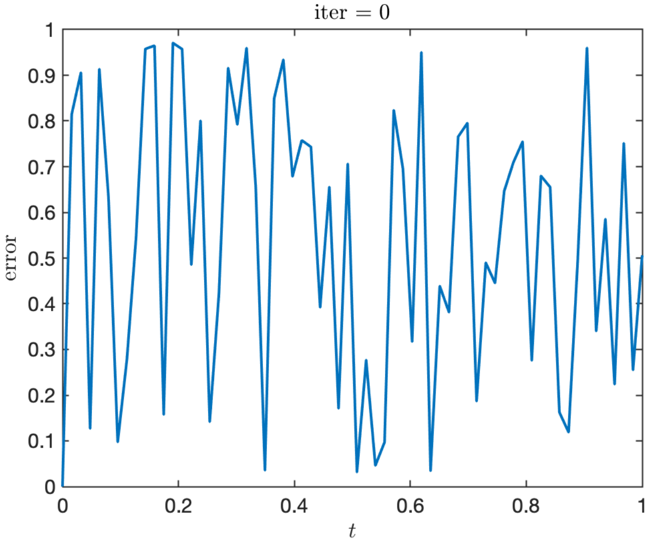

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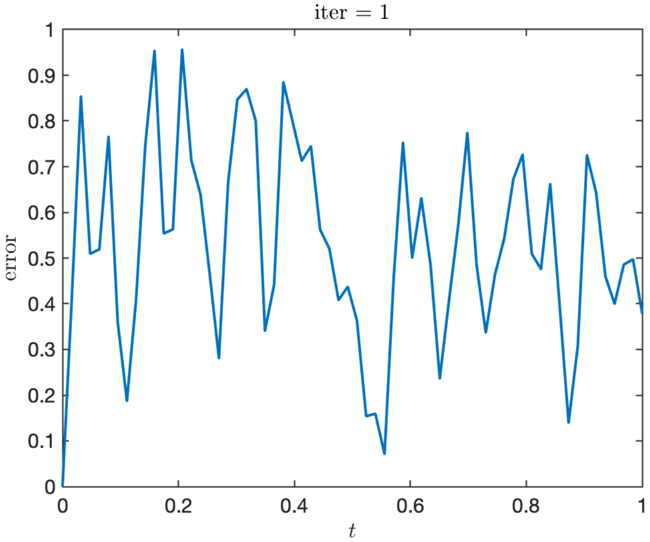

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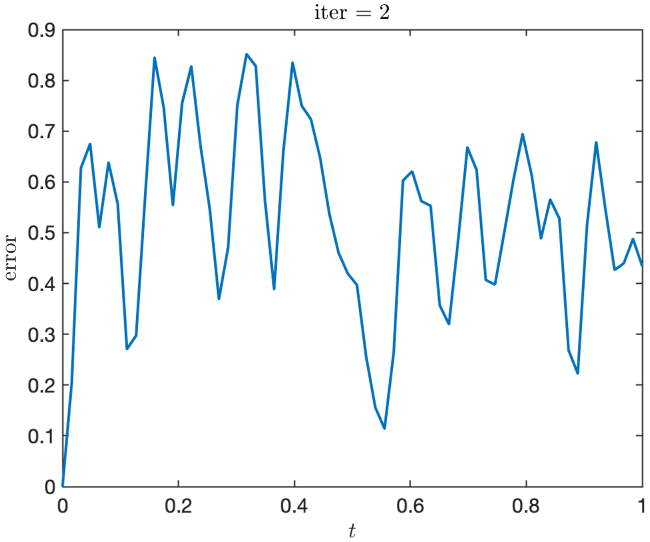

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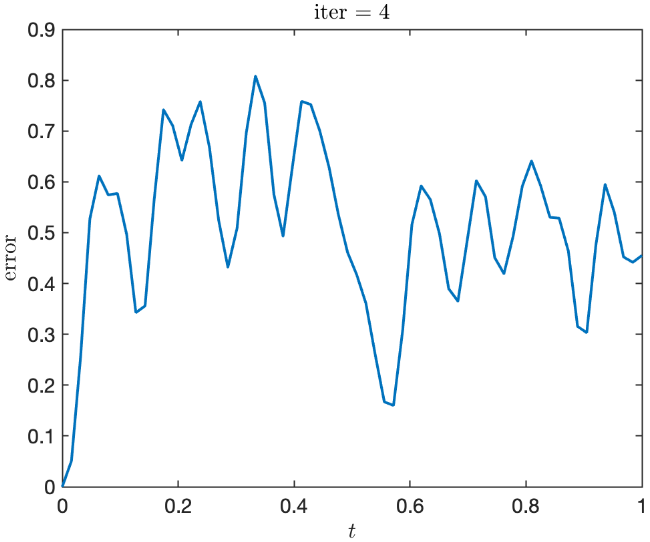

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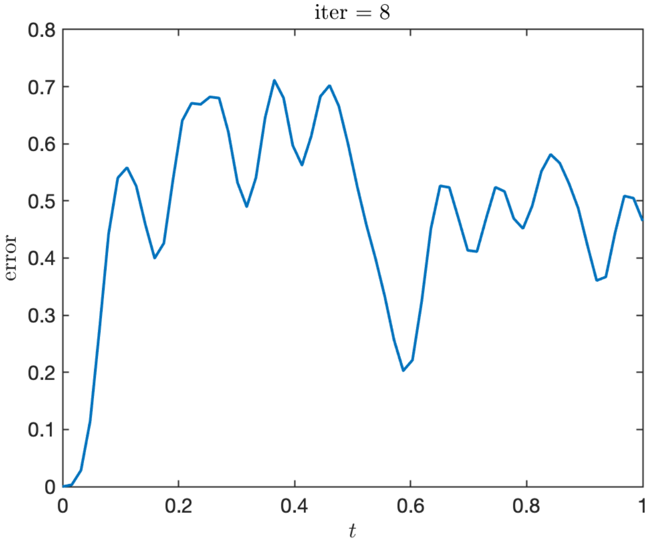

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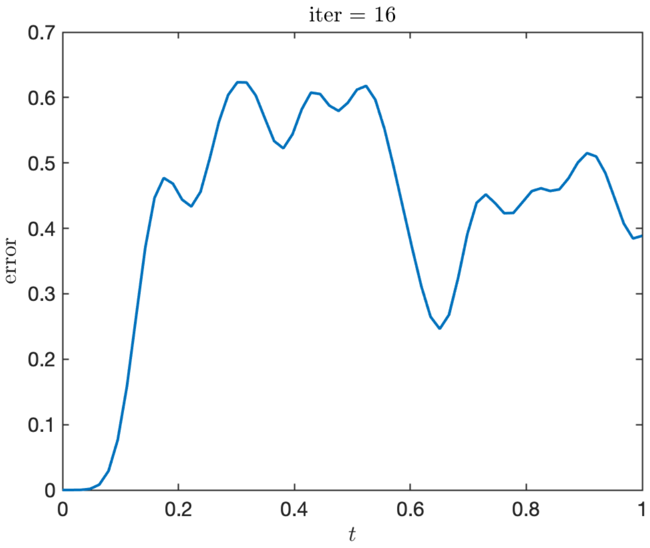

# Time Multigrid

★ **Dahlquist Equation**:

$$\partial_t u = \lambda u, \qquad u(0) = 0, \qquad \lambda \in \mathbb{C}.$$

★ **Discretization**: backward Euler

$$(1 - \Delta t \lambda) u_{n+1} - u_n = 0.$$

★ **Smoother**: damped Jacobi

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \alpha D^{-1}(\mathbf{b} - A\mathbf{u}^k) = \mathbf{u}^k - \frac{\alpha}{1 - \lambda \Delta t} A\mathbf{u}^k.$$

★ **Two-grid Method**

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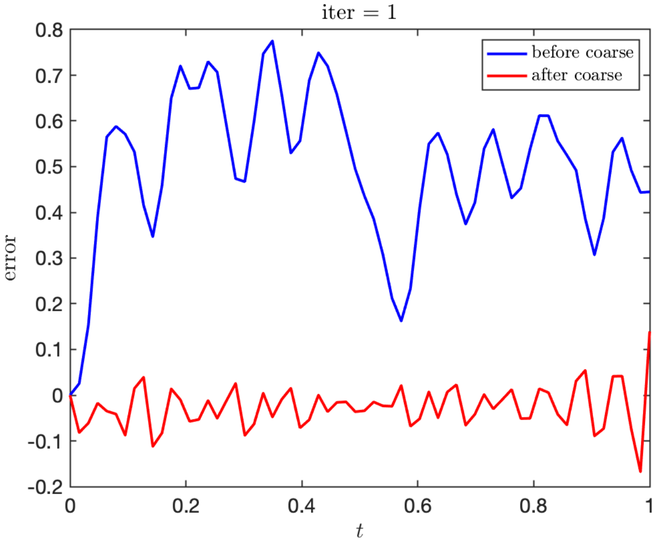

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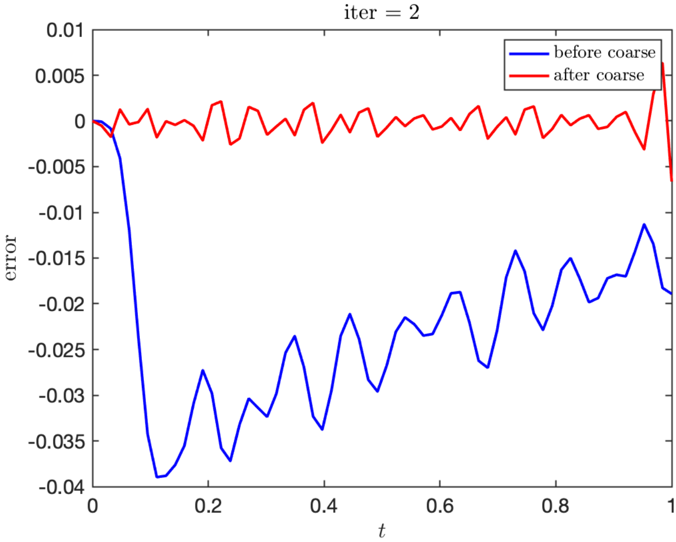

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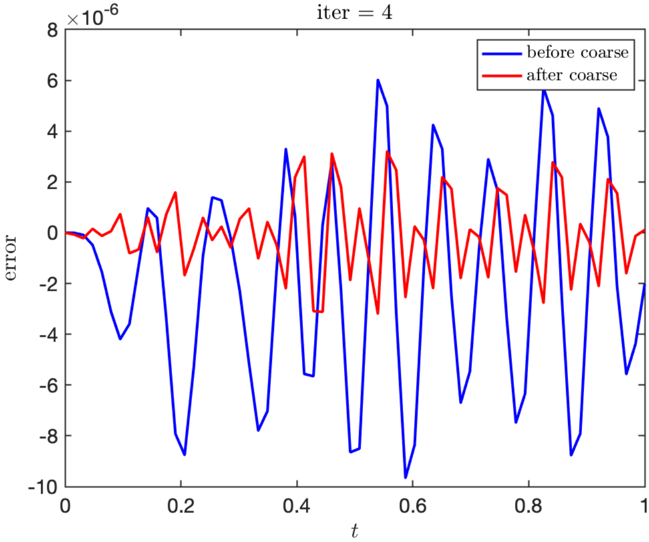

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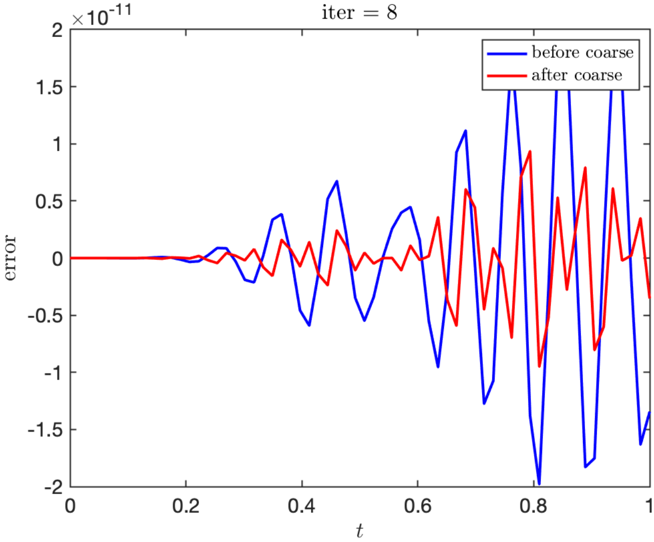

# Back to heat equation

★ **Dahlquist Equation**:

$$\partial_t u = \lambda u,$$

★ **Heat Equation**:

$$\partial_t u = \partial_{xx} u + f,$$

# Back to heat equation

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$$\partial_t u = \lambda u,$$

★ **Discretization**:

$$(1 - \Delta t \lambda) u_{n+1} - u_n = 0,$$

★ **Heat Equation**:

$$\partial_t u = \partial_{xx} u + f,$$

★ **Discretization**:

$$(I - \Delta t L)\mathbf{u}_{n+1} - \mathbf{u}_n = \Delta t \mathbf{f}_{n+1},$$

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★ **Heat Equation**:

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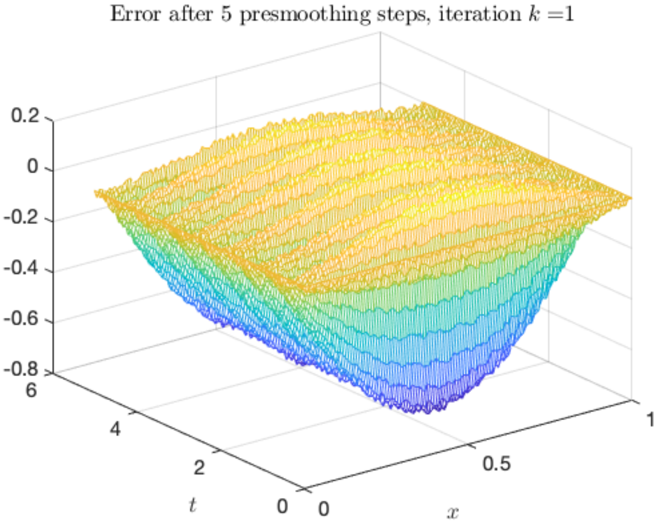
# Back to heat equation

**Settings**: $\nu_1 = \nu_2 = 5$ and damping parameter $\alpha = \frac{1}{2}$.



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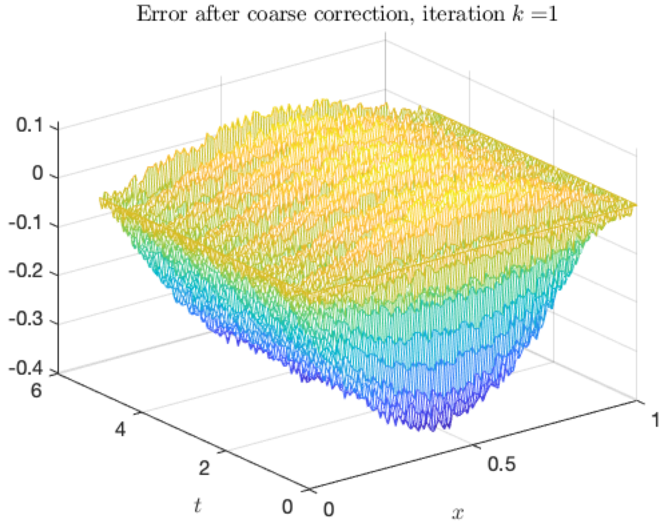

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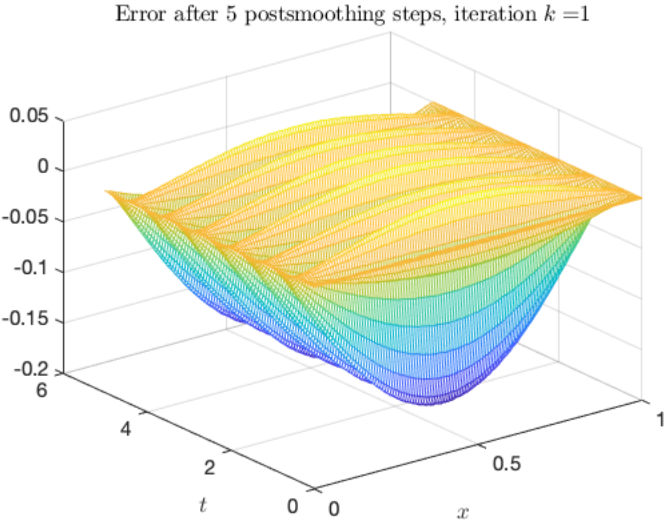

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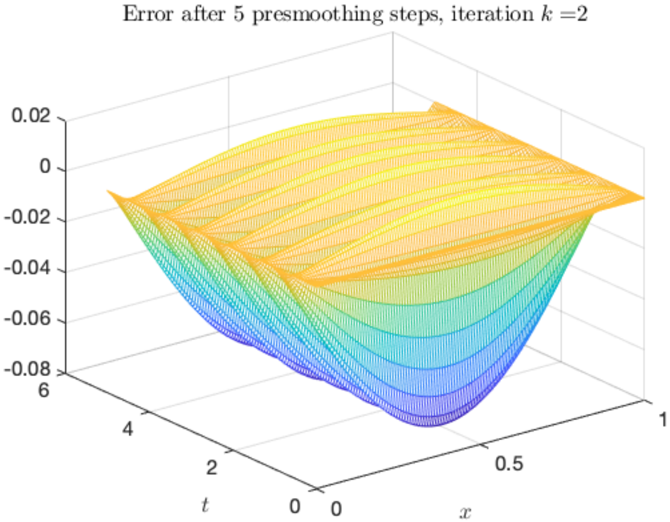

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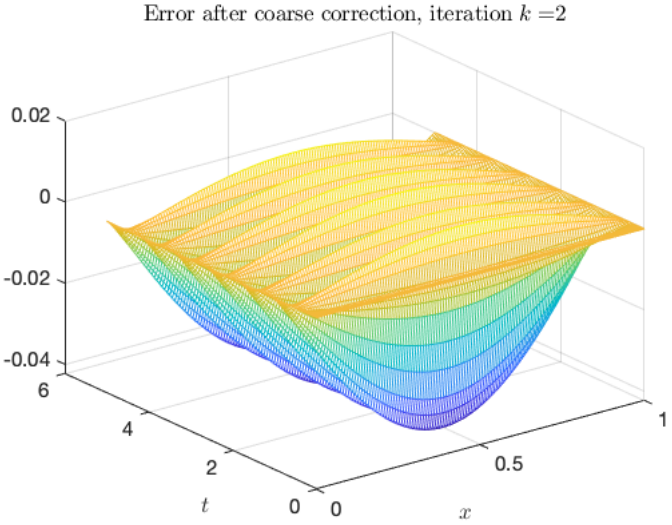

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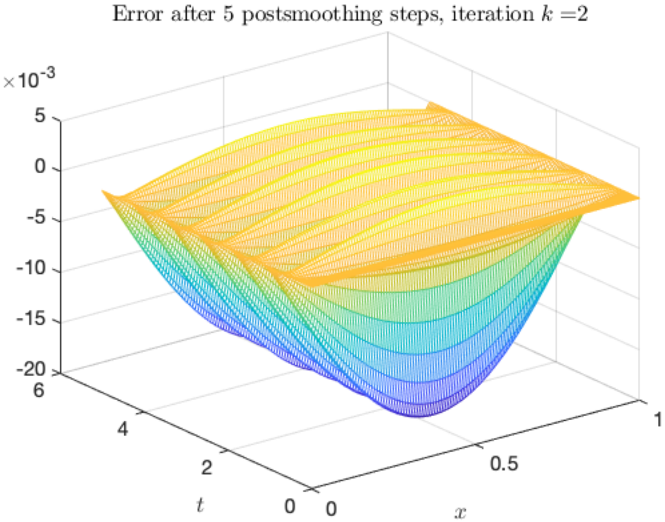

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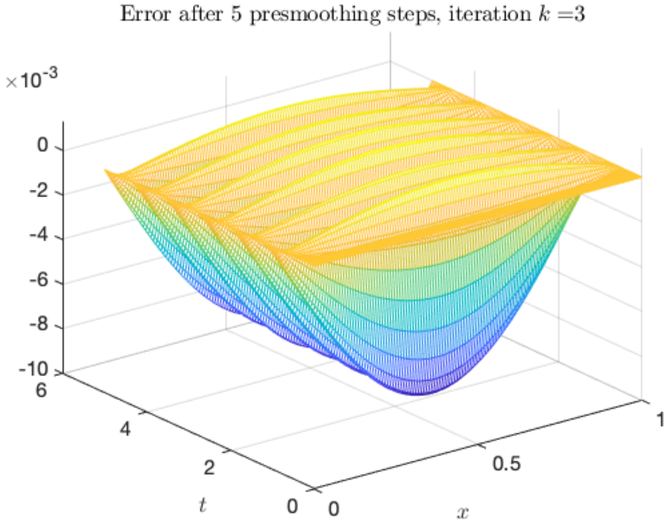

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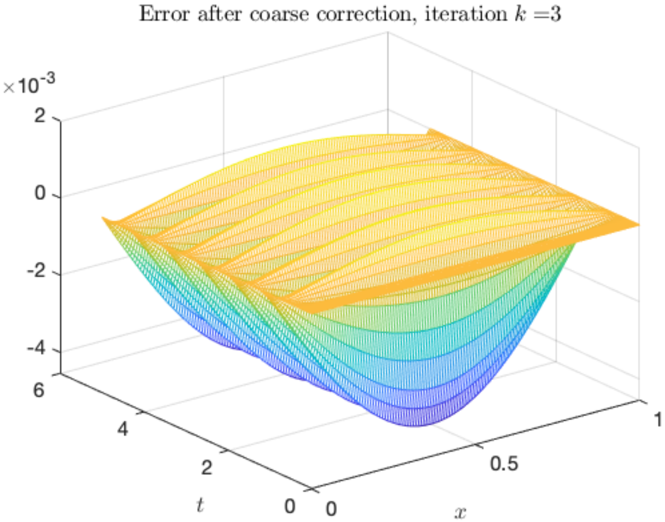

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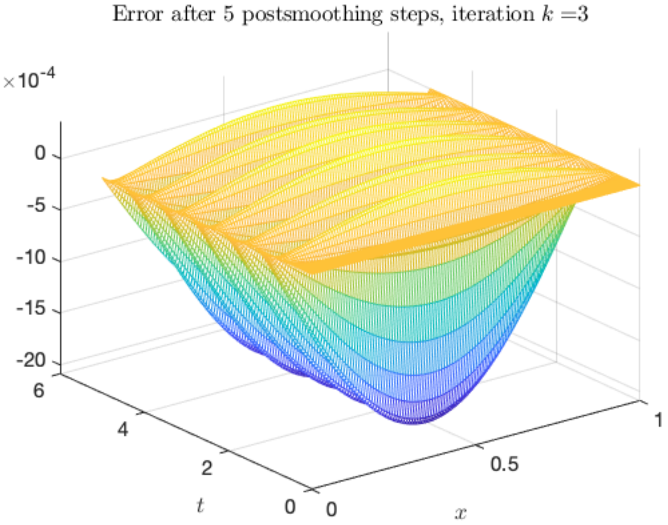

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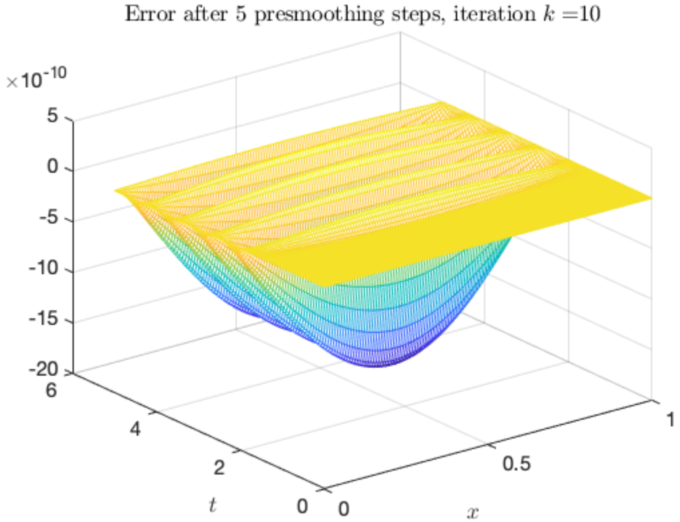

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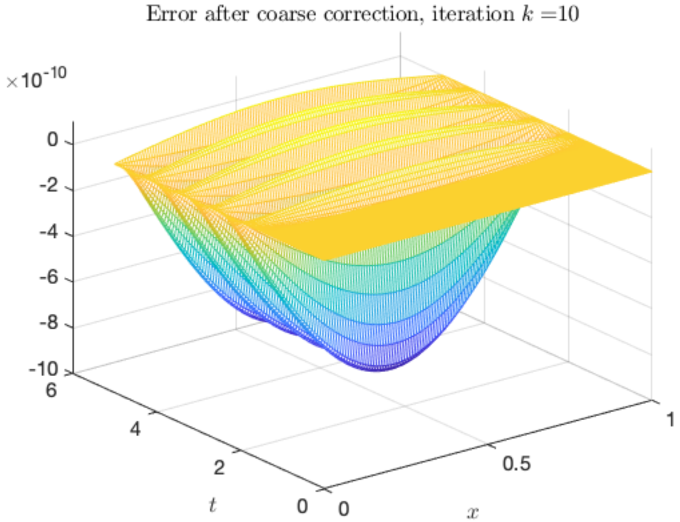

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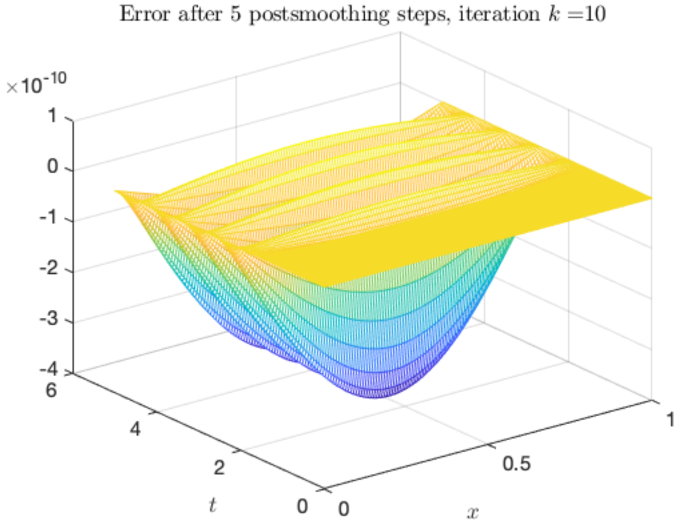

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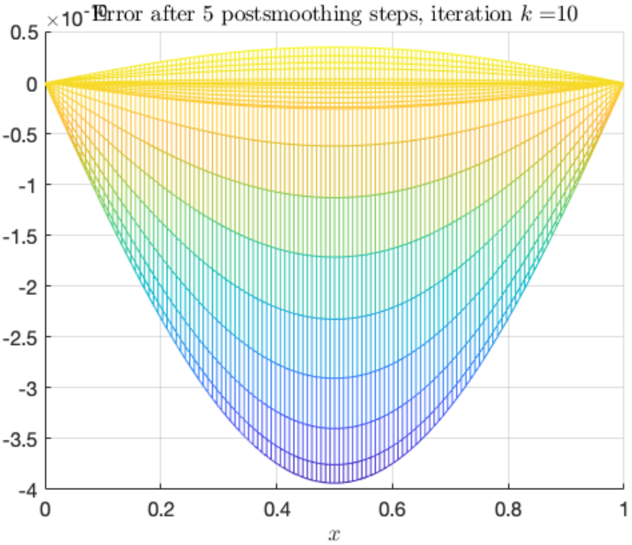

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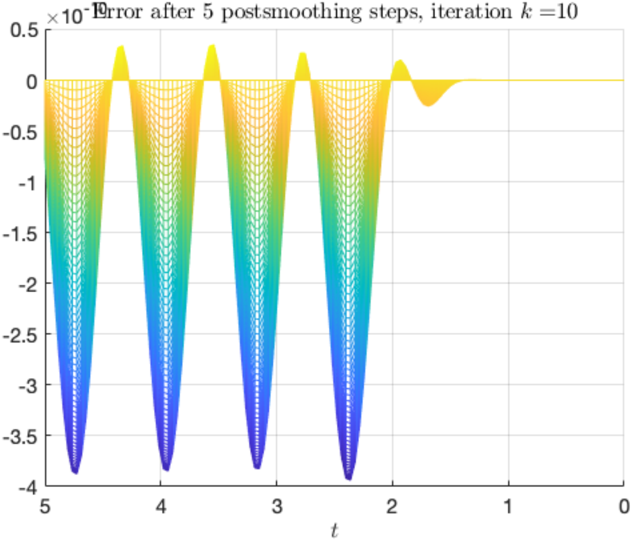

# Back to heat equation

★ **Heat Equation**:

$$\partial_t u = \partial_{xx} u + f,$$

★ **Discretization**:

$$(I - \Delta t L)\mathbf{u}_{n+1} - \mathbf{u}_n = \Delta t \mathbf{f}_{n+1},$$

★ **Smoother**:

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \alpha(I - \Delta t L)^{-1}(\mathbf{b} - A\mathbf{u}^k),$$

with

$$A = \begin{pmatrix} I - \Delta t L & & & \\ -I & I - \Delta t L & & \\ & -I & I - \Delta t L & \\ & & \ddots & \ddots \end{pmatrix}.$$

# Local Fourier Analysis

- Damped block Jacobi smoother:

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \alpha(I - \Delta t L)^{-1}(\mathbf{b} - A\mathbf{u}^k).$$

- Take the Fourier mode:

$$u_{j,n}^k = c_{\omega,\xi}^k e^{i\omega j \Delta x} e^{i\xi n \Delta t}.$$

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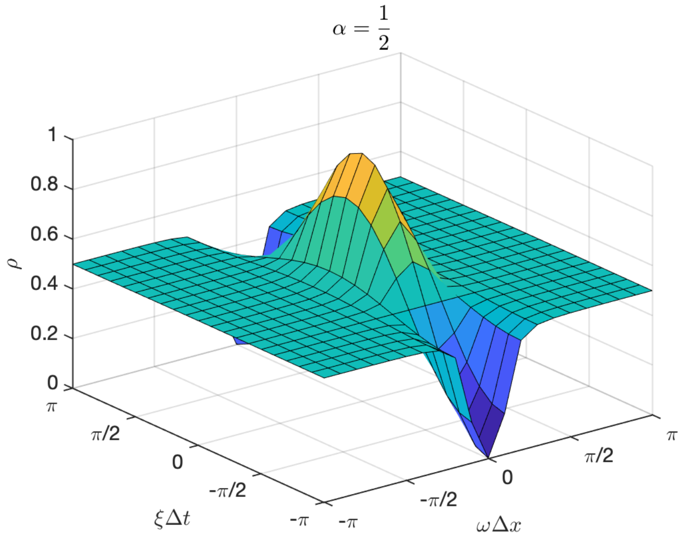
- Convergence factor:

$$\rho(\alpha, \omega, \xi) = 1 - \alpha(1 - \frac{e^{-i\xi\Delta t}}{1 - 2\frac{\Delta t}{\Delta x^2}(\cos(\omega \Delta x) - 1)}).$$

# Local Fourier Analysis



# Parabolic control

★ **Heat equation**:

$$\begin{aligned} \partial_t u - \Delta_x u &= z \quad && \text{in } Q, \\ u &= 0 \quad && \text{on } \Sigma, \\ u &= u_0 \quad && \text{on } \Sigma_0, \end{aligned}$$

$Q := (0, T) \times \Omega$, $\Sigma := (0, T) \times \partial\Omega$, $\Sigma_0 := \{0\} \times \Omega$ and $\Omega \subset \mathbb{R}^n$.

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★ **Cost functional**:

$$J(u, z) = \frac{1}{2}\|u - \hat{u}\|^2_{L^2(Q)} + \frac{\nu}{2}\|z\|^2_{L^2(Q)},$$

with $\nu > 0$.

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★ **Problem**: Find

$$\min_{z \in L^2(Q)} J(u, z),$$

subject to the PDE constraint.

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★ **Approach**: Lagrange multiplier $\lambda$.

# Optimality system

- First-order optimality system (forward-backward):

$$\begin{aligned} \partial_t u - \Delta_x u &= z && \text{in } Q, & -\partial_t \lambda - \Delta_x \lambda &= u - \hat{u} && \text{in } Q, \\ u &= 0 && \text{in } \Sigma, & \lambda &= 0 && \text{in } \Sigma, \\ u &= u_0 && \text{in } \Sigma_0, & \lambda &= 0 && \text{in } \Sigma_T, \end{aligned}$$

$$-\lambda + \nu z = 0 \quad \text{in } Q.$$

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$$-\lambda + \nu z = 0 \quad \text{in } Q.$$

- All-in-one system:

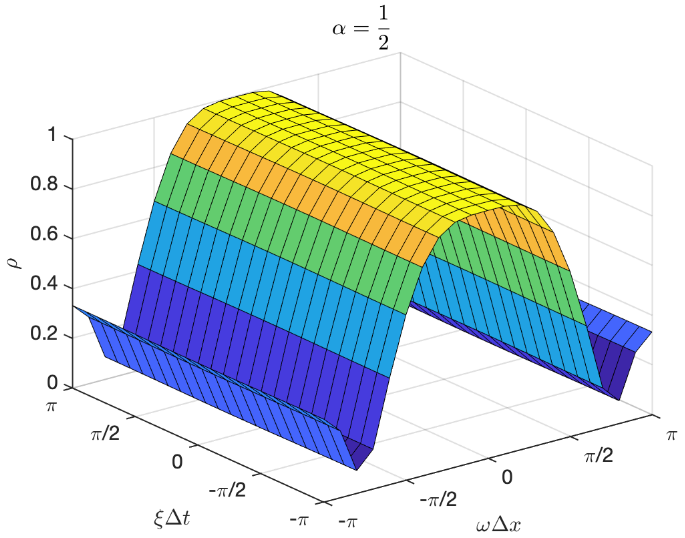
$$\begin{aligned} \partial_{tt} u - \Delta_x^2 u &= \frac{u - \hat{u}}{\nu} && \text{in } Q, \\ \partial_t u - \Delta_x u &= 0 && \text{in } \Sigma, \\ u &= 0 && \text{in } \Sigma, \\ u &= u_0 && \text{in } \Sigma_0, \\ \partial_t u - \Delta_x u &= 0 && \text{in } \Sigma_T. \end{aligned}$$

# Local Fourier Analysis

Convergence factor for damped Jacobi smoother:

$$\rho(\alpha,\omega,\xi) = 1 - \alpha\left(1 - \frac{\frac{2\cos(\xi\Delta t)}{\Delta t^2} - \frac{-8\cos(\omega\Delta x)+2\cos(2\omega\Delta x)}{\Delta x^4}}{\frac{6}{\Delta x^4} + \frac{2}{\Delta t^2} + \frac{1}{\nu}}\right).$$

# Local Fourier Analysis

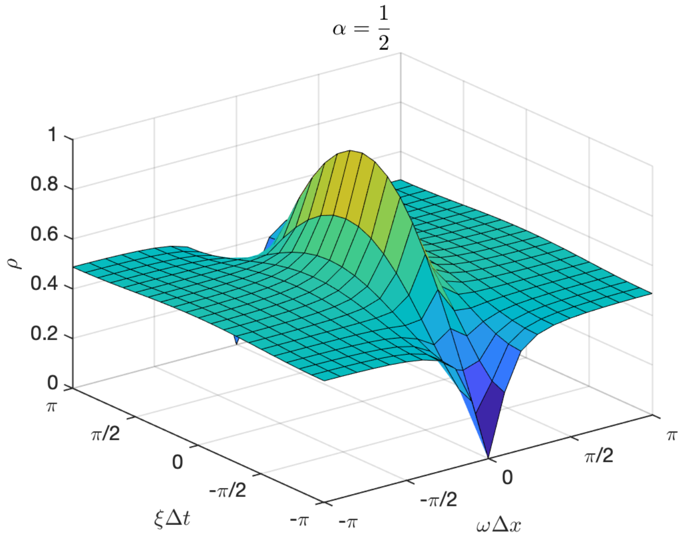


# Local Fourier Analysis

Convergence factor for damped block Jacobi smoother:

$$\rho(\alpha, \omega, \xi) = 1 - \alpha\left(1 - \frac{2\cos(\xi \Delta t)}{2 + \Delta t^2\left(\frac{2\cos(2\omega\Delta x) - 8\cos(\omega\Delta x) + 6}{\Delta x^4} + \frac{1}{\nu}\right)}\right).$$

# Local Fourier Analysis



# Local Fourier Analysis

**Parabolic**:

Damped Jacobi: $\rho(\alpha,\omega,\xi) := 1-\alpha(1-\frac{2\frac{\Delta t}{\Delta x^2}\cos(\omega\Delta x)+e^{-i\xi\Delta t}}{1+2\frac{\Delta t}{\Delta x^2}})$.

Damped block Jacobi: $\rho(\alpha,\omega,\xi) = 1-\alpha(1-\frac{e^{-i\xi\Delta t}}{1+2\frac{\Delta t}{\Delta x^2}(1-\cos(\omega\Delta x))})$.

**Parabolic control**:

Damped Jacobi:

$$\rho(\alpha,\omega,\xi) = 1-\alpha(1-\frac{\frac{2\cos(\xi\Delta t)}{\Delta t^2}-\frac{2\cos(2\omega\Delta x)-8\cos(\omega\Delta x)}{\Delta x^4}}{\frac{6}{\Delta x^4}+\frac{2}{\Delta t^2}+\frac{1}{\nu}}).$$

Damped block Jacobi:

$$\rho(\alpha,\omega,\xi) = 1-\alpha(1-\frac{2\cos(\xi\Delta t)}{2+\Delta t^2(\frac{2\cos(2\omega\Delta x)-8\cos(\omega\Delta x)+6}{\Delta x^4}+\frac{1}{\nu})}).$$

Thanks for your attention !