# Some modelling and optimization problems for microalgal raceway ponds

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# Overview

# Motivation and Framework

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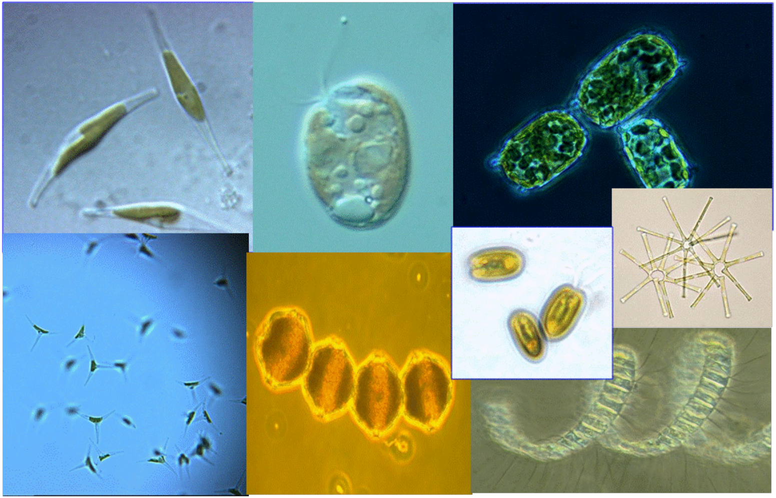
  - photosynthetic micro-organisms,
  - 2 to 50 micro-meters,



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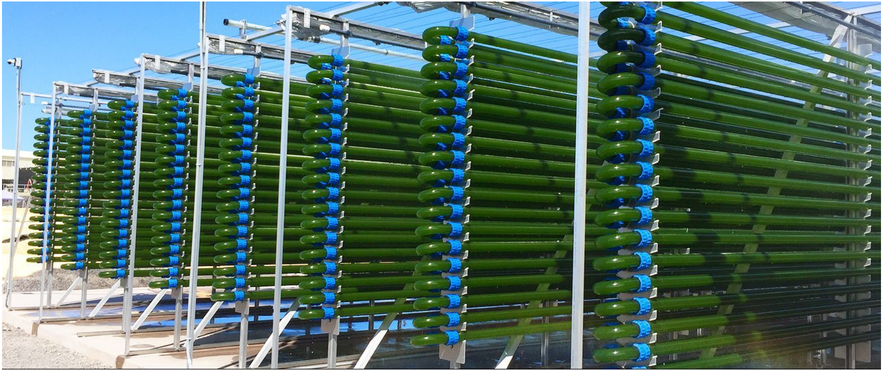

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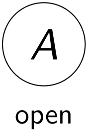

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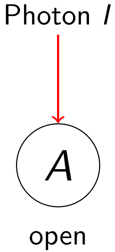

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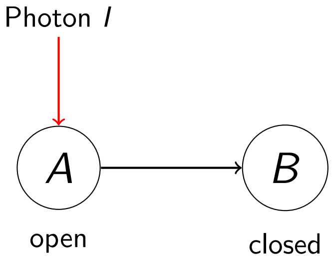

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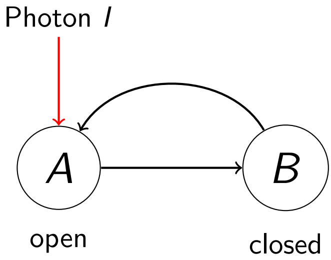

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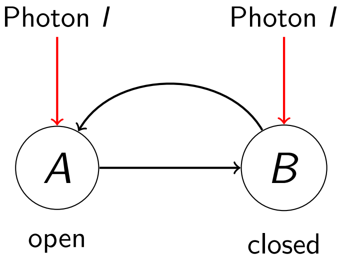

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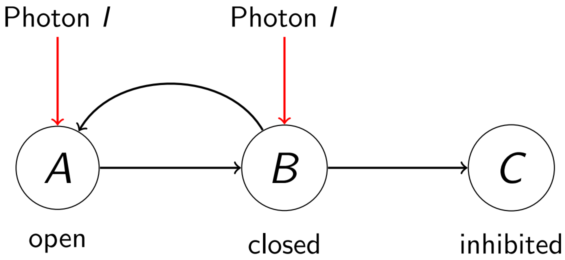

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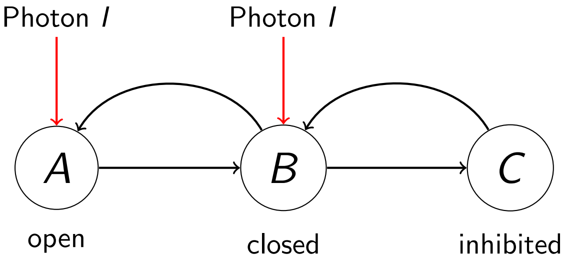

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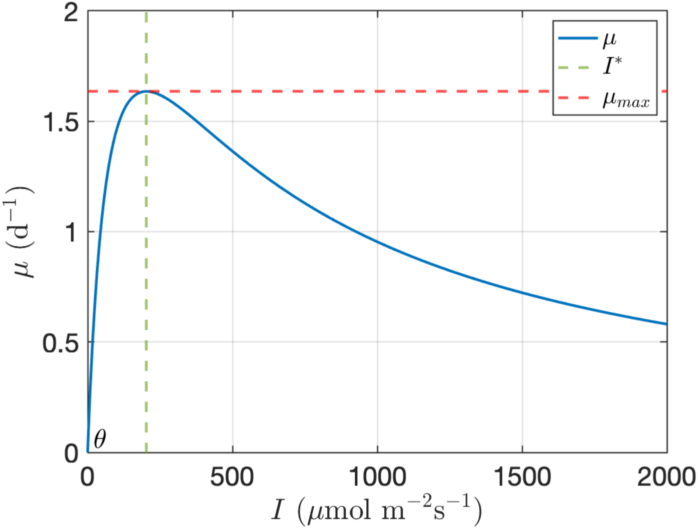
Haldane description



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- The Beer-Lambert law: $I(z) = I_s \exp\Big( -\varepsilon z \Big)$.

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- widely used and cheapest cultivation system,
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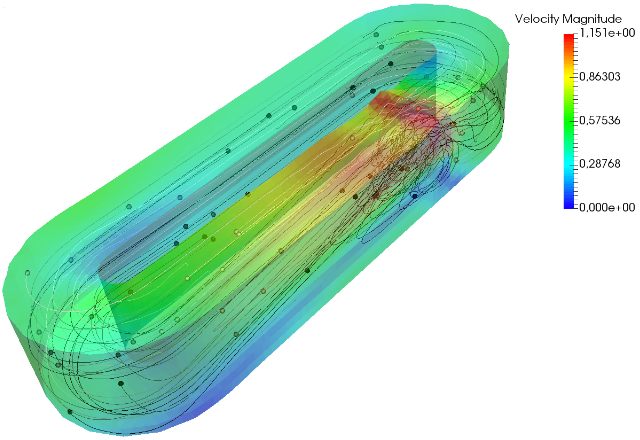
# Raceway modelling

Simulation of the trajectories with the code FreshKiss3D (*Demory et al.* 2018).



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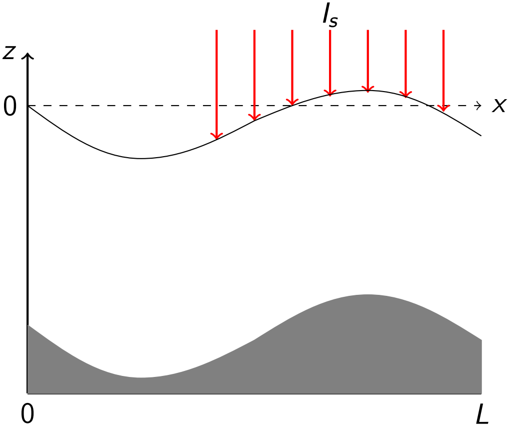


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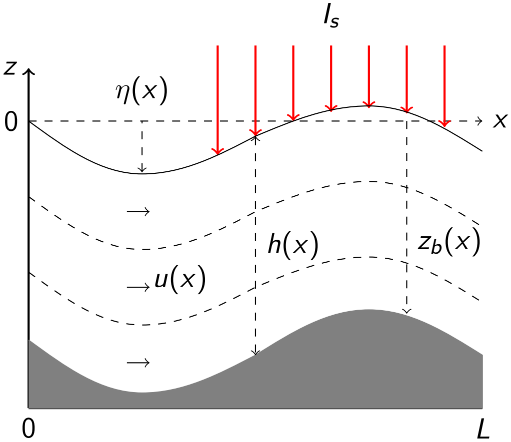

# Saint-Venant Equations

- 1D steady state Saint-Venant equations

$$\partial_x(hu) = 0, \quad \partial_x(hu^2 + g\frac{h^2}{2}) = -gh\partial_x z_b.$$

# Saint-Venant Equations

- Relation between $z_b$ and $h$

$$z_b = \frac{M_0}{g} - \frac{Q_0^2}{2gh^2} - h, \tag{1}$$

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- A **time free** formulation of the Lagrangian trajectory starting from $z(0)$:

$$z(x) = \eta(x) + \frac{h(x)}{h(0)}(z(0) - \eta(0)). \tag{2}$$

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- Adjoint method $\to$ $\nabla\bar{\mu}_{N_z}(a)$.

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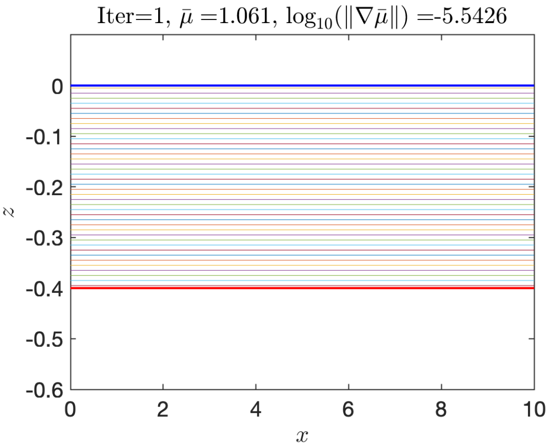
- Number of trajectories: $N_z = 40$.
- Initial guess: flat topography.



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**Theorem (Flat topography)**

*Assume the volume of the system $V$ is constant. Then $\nabla \bar{\mu}_{N_z}(0) = 0$.*

# Optimal topography ($C$ periodic)

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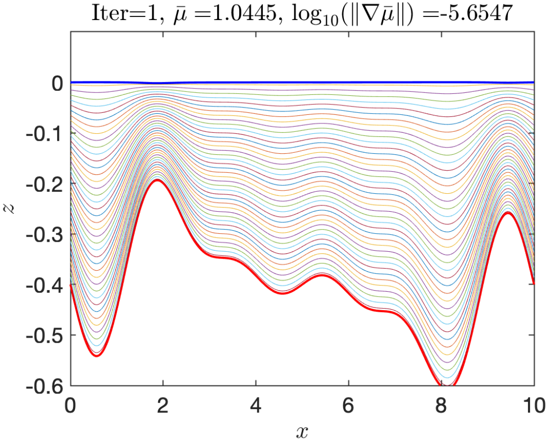

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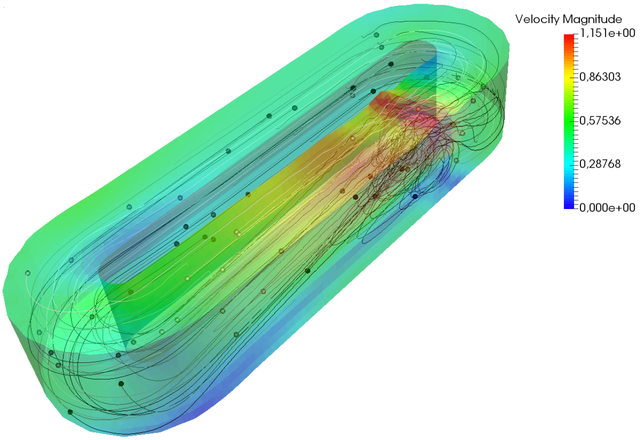
- In the case $C$ periodic, the flat topography is not only a critical point but also the optimal topography.
- What can be further optimized?

# Overview

# Mixing devices

Simulation of the trajectories with the code FreshKiss3D (*Demory et al.* 2018).



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**Notations**

We denote by $\mathcal{P}$ the set of **permutation matrices** of size $N_z \times N_z$ and by $\mathfrak{S}_{N_z}$ the associated set of permutations of $N_z$ elements.

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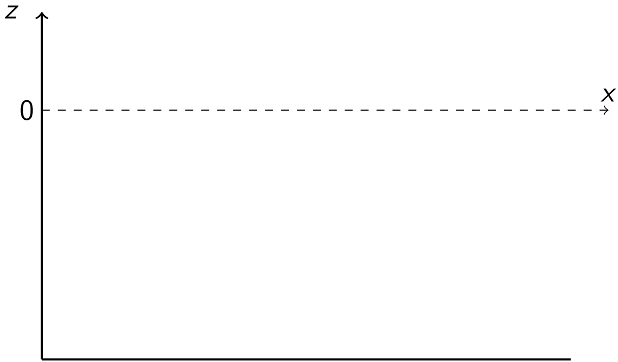

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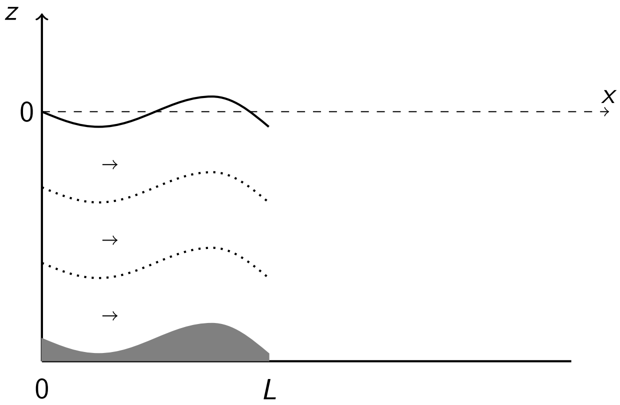

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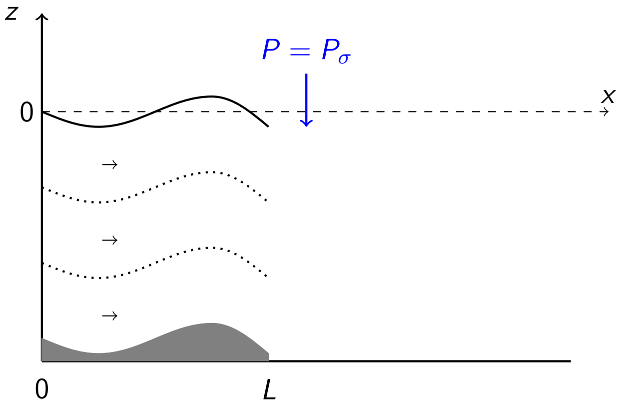

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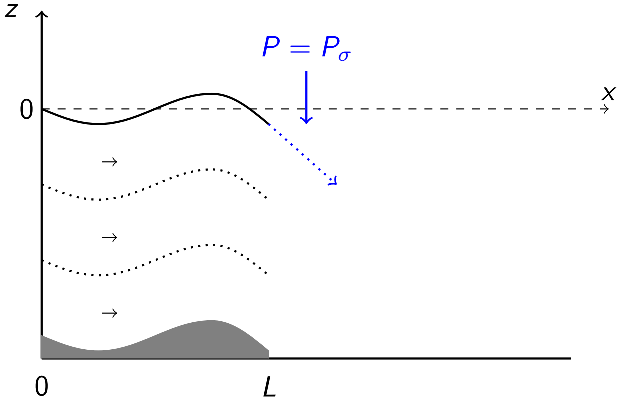

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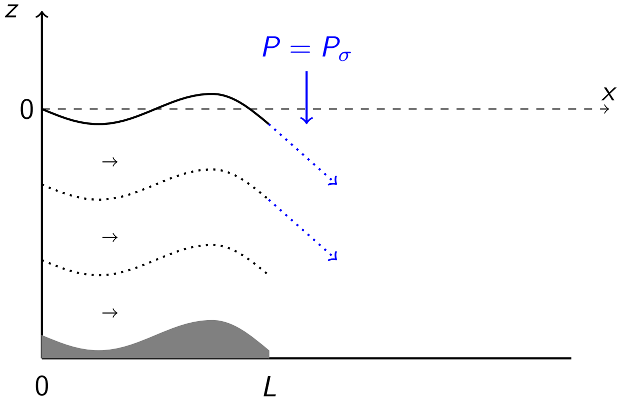

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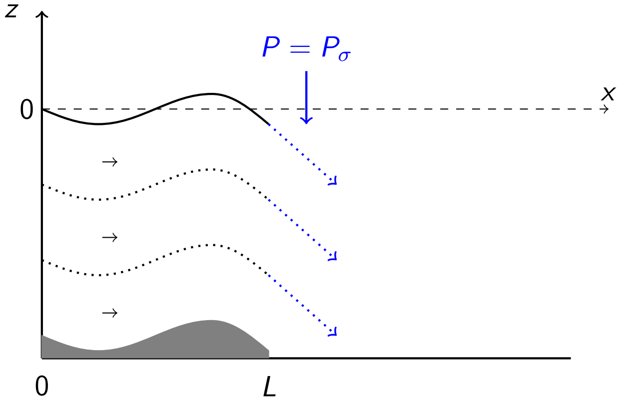

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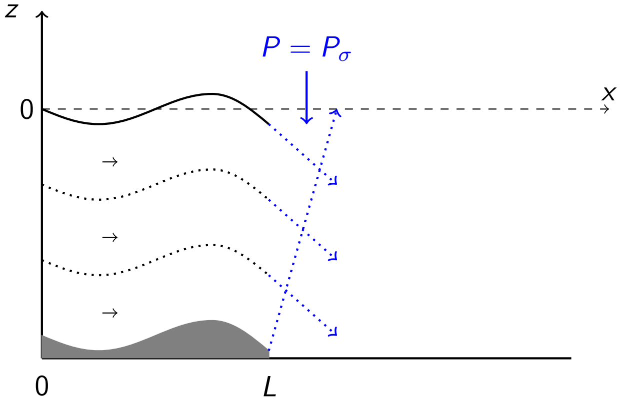

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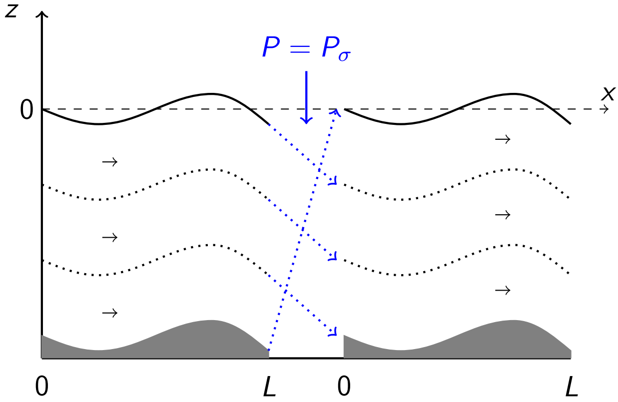

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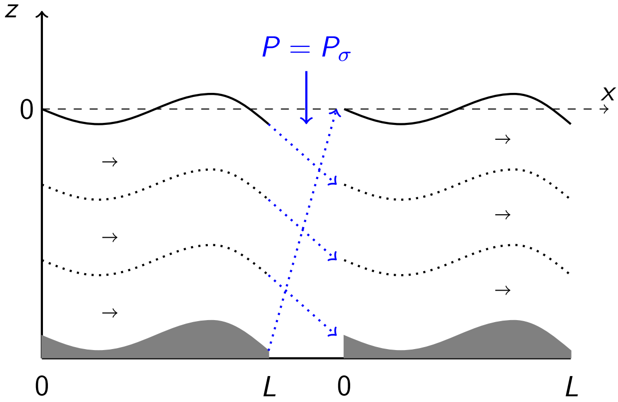


- Choice of Period?

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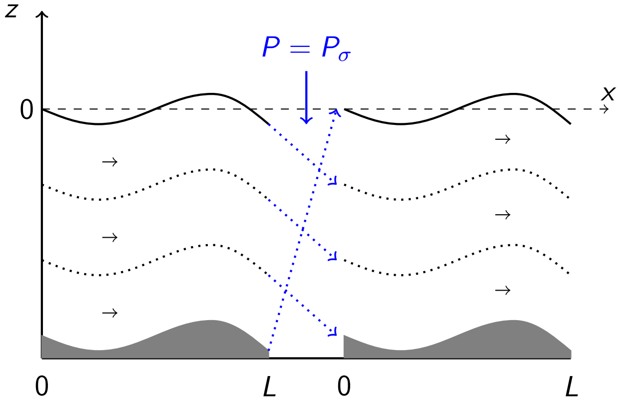


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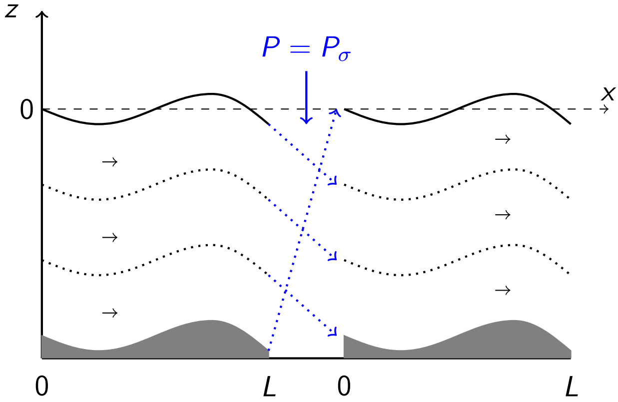
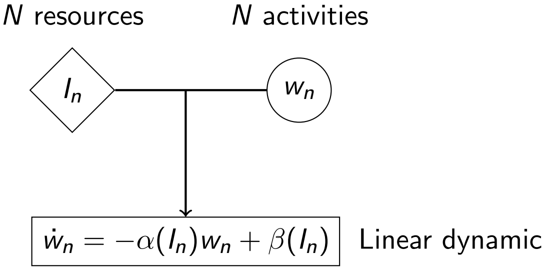



- Choice of Period? Order of $\sigma$.
- Re-distribution of light.

# Periodic dynamical resource allocation problem

$N$ resources $N$ activities

# Periodic dynamical resource allocation problem



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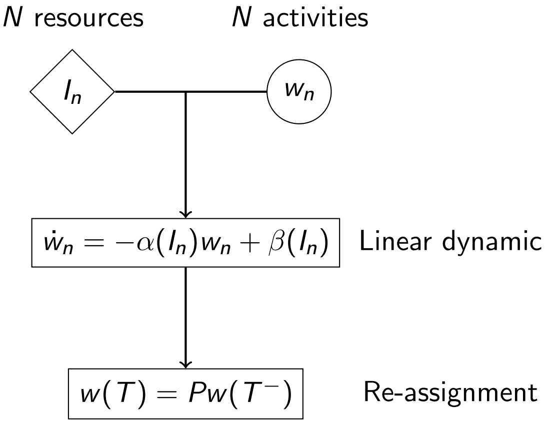

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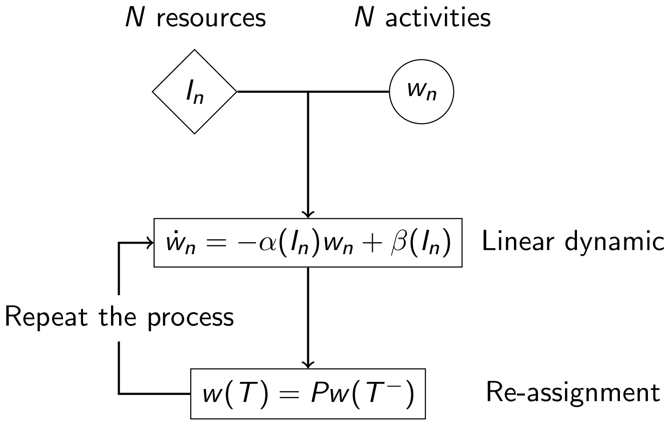

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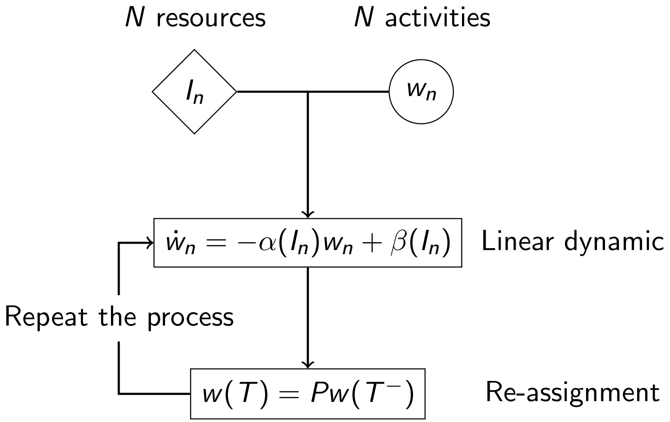


**Theorem (One period is enough)**

*If $w$ is $KT$-periodic (i.e., $w(T_K) = w(T_0)$), then $w$ is $T-$periodic.*

# Original problem

## Optimization problem

$$\max_{P \in \mathcal{P}} J(P) := \max_{P \in \mathcal{P}} \langle u, (\mathcal{I}_N - PD)^{-1} P v \rangle, \tag{3}$$

Two vectors $u, v$ and a diagonal matrix $D$ all depend on $(I_n)_{n=1}^N$.

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Two vectors $u, v$ and a diagonal matrix $D$ all depend on $(I_n)_{n=1}^N$.

## Remark

*Since $\#\mathfrak{S} = N!$, this problem cannot be tackled in realistic cases where large values of $N$ must be considered, e.g., to keep a good numerical accuracy.*

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## Optimization problem

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Two vectors $u, v$ and a diagonal matrix $D$ all depend on $(l_n)_{n=1}^N$.

## Remark

*Since $\#\mathfrak{S} = N!$, this problem cannot be tackled in realistic cases where large values of N must be considered, e.g., to keep a good numerical accuracy.*

Expand the functional (3) as follows

$$\underbrace{\langle u, (\mathcal{I}_N - PD)^{-1} Pv \rangle}_{J(P)} = \sum_{\ell=0}^{+\infty} \langle u, (PD)^\ell Pv \rangle = \underbrace{\langle u, Pv \rangle}_{J^{\text{approx}}(P)} + \sum_{\ell=1}^{+\infty} \langle u, (PD)^\ell Pv \rangle,$$

## Simplified problem

$$\max_{P \in \mathcal{P}} J^{\text{approx}}(P) := \max_{P \in \mathcal{P}} \langle u, Pv \rangle. \tag{4}$$

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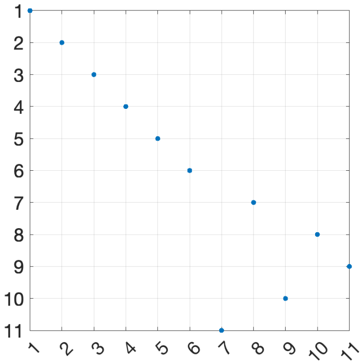
## Lemma (Optimal matrix)

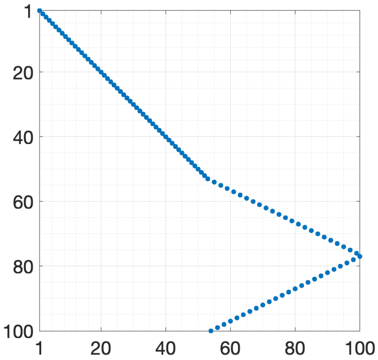
- *$P_+$: associates the largest coefficient of $u$ with the largest coefficient of $v$, the second largest coefficient with the second largest, and so on.*
- *$P_-$: associates the largest coefficient of $u$ with the smallest coefficient of $v$, the second largest coefficient with the second smallest, and so on.*

# Optimal Matrix

Test for $(I_s, q, T) = (2000, 5\%, 1000)$.



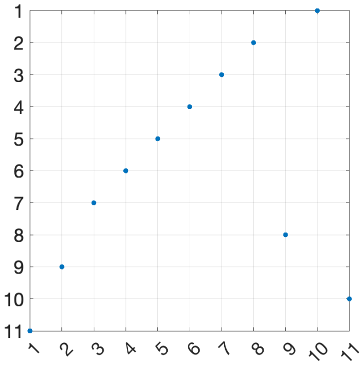

$P_{\max}$ for $J(P)$




$P_+$ for $J^{\text{approx}}(P)$

# Optimal Matrix

Test for $(I_s, q, T) = (800, 0.5\%, 1)$.

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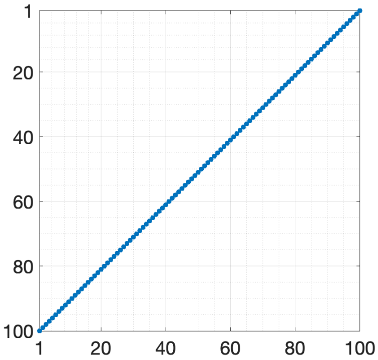



$P_+$ for $J^{\text{approx}}(P)$

# Quality of the approximation

## Theorem (Coincidence Criterion: $P_{\max} = P_+$?)

*Assume that $u$ and $v$ have positive entries and define*

$$\phi(m) := \frac{1}{s_{\lceil \frac{m}{2} \rceil}} \left( \sum_{\ell=1}^{+\infty} d_{\max}^{\ell} F^{+}_{(\ell+1)m} - d_{\min}^{\ell} F^{-}_{(\ell+1)m} \right), \tag{5}$$

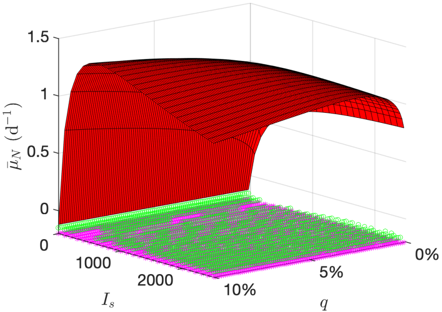
*where* $m := \# \{ n = 1, \ldots, N \mid \sigma(n) \neq \sigma_+(n) \}$, $d_{\max} := \max_{n=1,\ldots,N}(d_n)$ *and* $d_{\min} := \min_{n=1,\ldots,N}(d_n)$. *Assume that:*

$$\max_{m \geq 2} \phi(m) \leq 1. \tag{6}$$

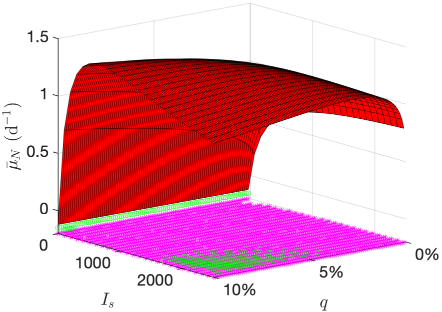
*Then* $P_{\max} = P_+$.

# Approximation and criterion

$T = 1000$.




$N = 5$

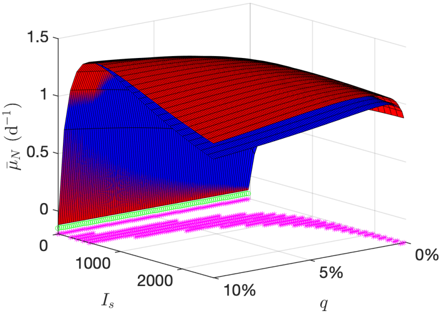



$N = 9$

- $\bar{\mu}_N(P_{\max})$ and $\bar{\mu}_N(P_+)$.
- $P_{\max} = P_+$.
- Coincidence Criterion satisfied.

# Approximation and criterion

$T = 1.$

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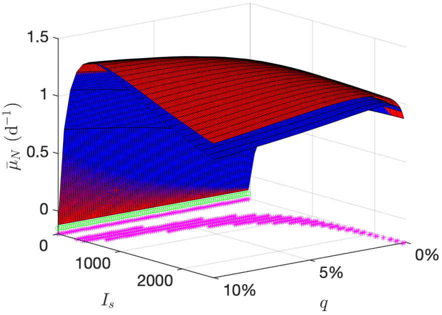


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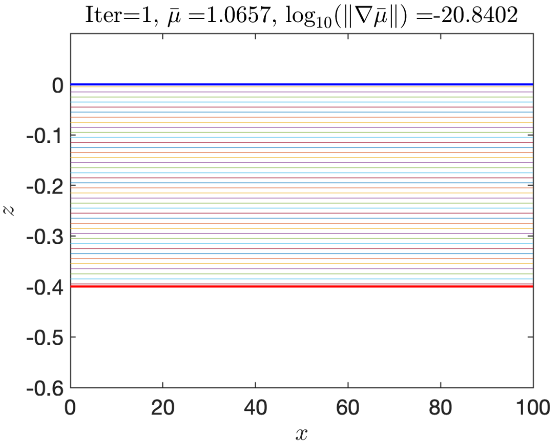
# Overview

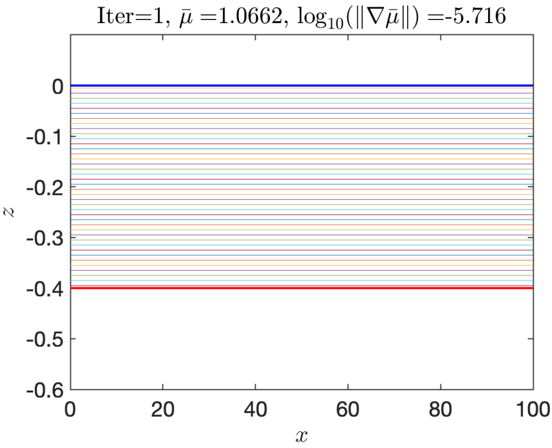
# Test with a permutation

We keep $N_a = 5$, $N_z = 40$ and choose $\sigma = Id$



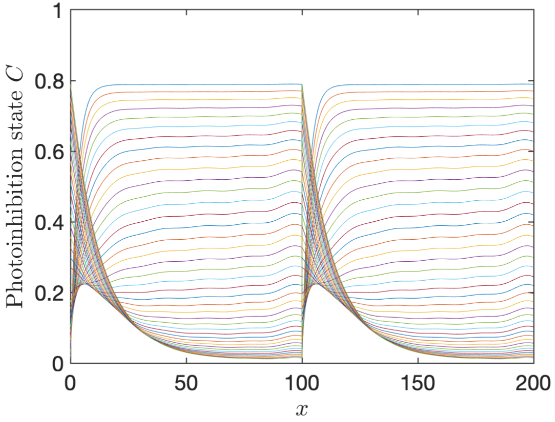
# Test with a permutation

- Test permutation: $\sigma = (1\ N_z)(2\ N_z - 1)\ldots$.
- Initial guess: flat topography.



# One periodic

We keep $N_a = 5$, $N_z = 40$ and choose $\sigma = (1\ N_z)(2\ N_z - 1)\ldots$



# Variable volume

- Volume related parameter $a_0$ as the average depth of the raceway system:

$$a_0 := \bar{h} = \frac{1}{L} \int_0^L h(x) \mathrm{d}x = \frac{V}{L}. \tag{7}$$

New parameter $\tilde{a} = [a_0, a_1, \ldots, a_{N_a}] \in \mathbb{R}^{N_a+1}$.

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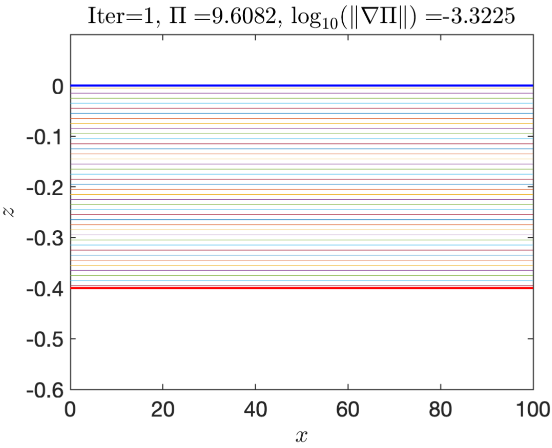
- Relation between $X$ and $V$: $Y_{\text{opt}}$.
- Optimization Problem:

$$\Pi_{N_z}(\tilde{a}) := \bar{\mu}_{N_z}(\tilde{a})Xh(\tilde{a}) = \frac{Y_{\text{opt}} - \alpha_1 a_0}{VN_z\alpha_0}\sum_{i=1}^{N_z}\int_0^L \mu(C_i, I_i(\tilde{a}))h(\tilde{a})\mathrm{d}x.$$

# Optimal Topography (Variable volume)

- Initial average depth: $a_0 = 0.4$m.
- Initial guess: flat topography.



# Optimal Topography (Variable volume)

- Number of trajectories: $N_z = 7$.
- Initial average depth: $a_0 = 0.4$m.
- Initial guess: flat topography.

$$P^{100}_{\max} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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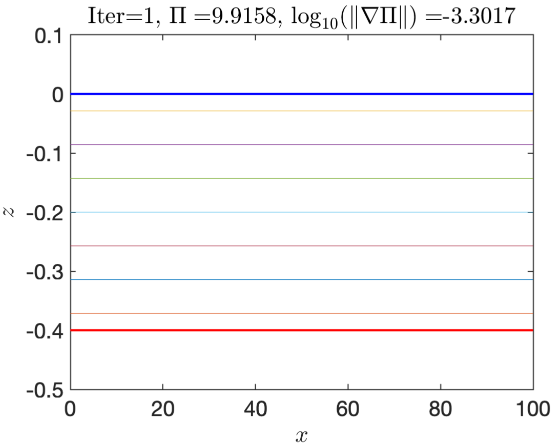
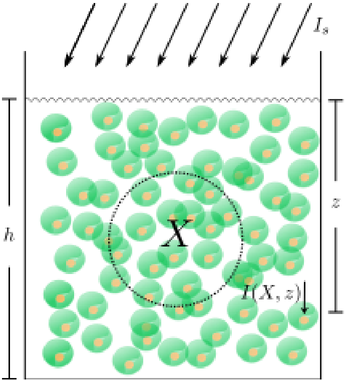


# Overview

# Maximize productivity

*Masci et al.* 2010:

- Growth $\mu$: Droop function.



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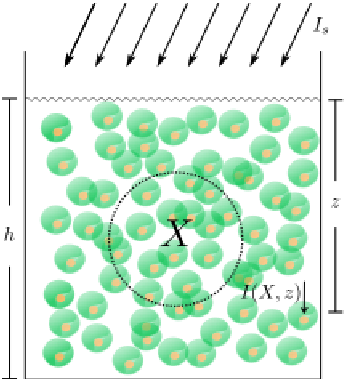

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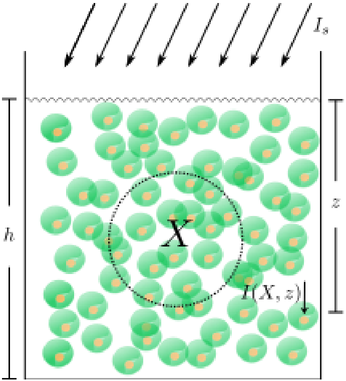

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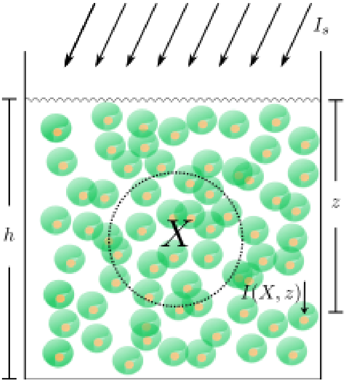

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- New concept: *optical depth productivity* $P := (\bar{\mu} - R)Y$ with the *optical depth* $Y := \varepsilon(X)h$.

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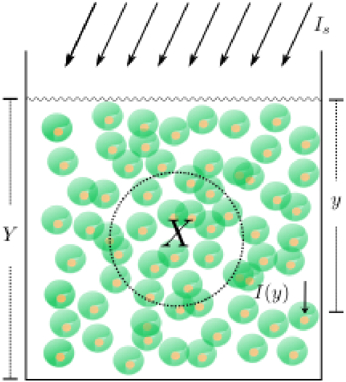

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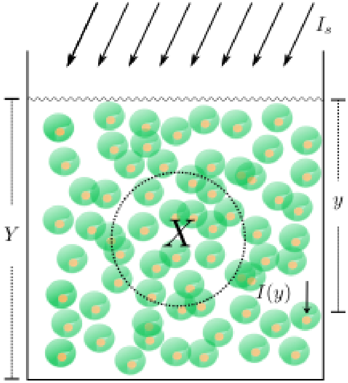

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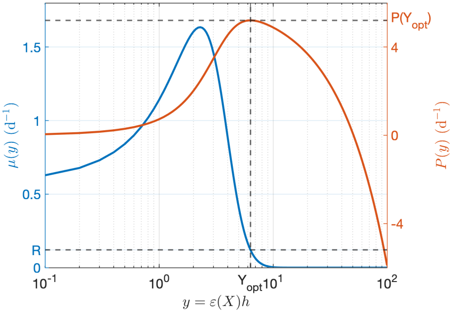

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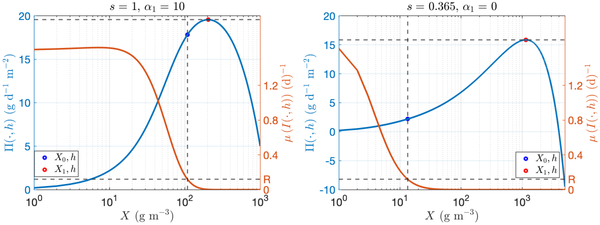
- The extinction function $\varepsilon(X) := \alpha_0 X^s + \alpha_1$ (*Morel* 1988, *Martínez* 2018).
- For a given depth $h$, $Y_{\mathrm{opt}}$ is generally NOT the optimal condition.



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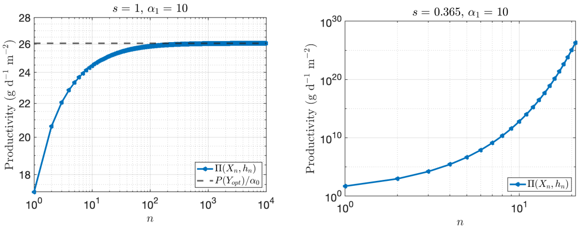
## Theorem

*If* $s = 1$, $\lim_{n\to\infty}\Pi(X_n, h_n) = \frac{P(Y_{opt})}{\alpha_0}$. *If* $s < 1$, $\lim_{n\to\infty}\Pi(X_n, h_n) = +\infty$.



# Nonlinear Controller

In real life application, $h$ is given, one would like to find $X_{\text{opt}}(h)$.

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## Proposition

*The control law*

$$D = \begin{cases} D_{\max} & X \geq \bar{X} \\ (\bar{\mu}(X,h) - R)\frac{X}{X^\star} & X < \bar{X} \end{cases}$$

*globally stabilizes the evolution of $X$ towards the positive point $X^\star$.*

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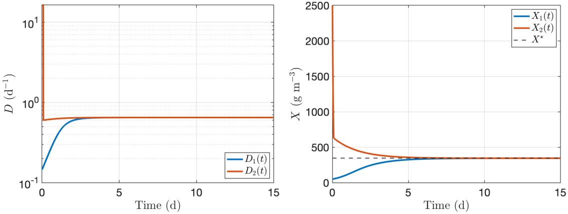

# Overview

# Conclusion

Topography:

- Flat topography is optimal in periodic case.
- Non flat topography with limited increase.

Mixing:

- Periodic dynamic resource allocation problem.
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| | Topography | Mixing | Depth / Biomass concentration |
|---|---|---|---|
| Gain | $\approx 1$ % | $\approx 30$ % | $\approx 100$ % |

# Future work

Further step that can lead to higher gains:

- Consider the turbulence regime (much more complex...).

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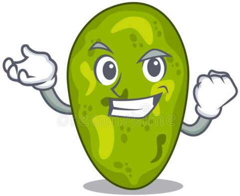
- Consider the turbulence regime (much more complex...).

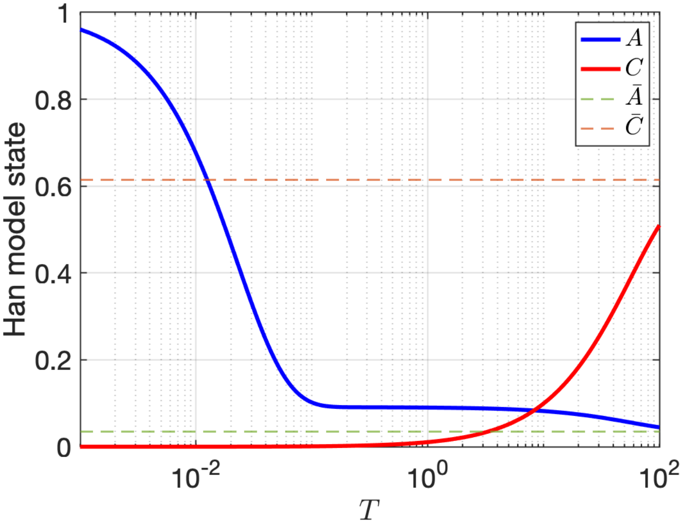
But for this:

- Include the faster time scales of the Han model.
- A more refined model of the mixing device (and its implication on hydrodynamics) must be developed.
- Higher energetic cost for maintaining a turbulent regime must be taken into account.

# Thanks for your attention

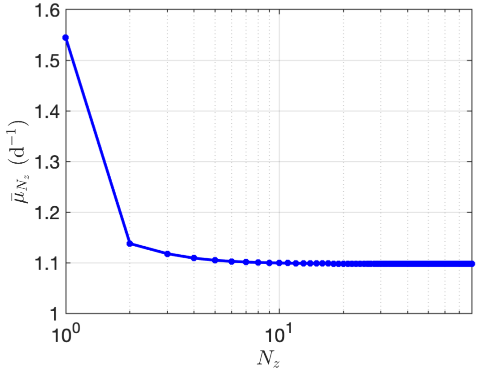


# Fast/slow illustration



# Effect on vertical discretization number

We fix $N_a = 5$ and take 100 random vector $a$. For $N_z$ varying from 1 to 80, we compute the average value of $\bar{\mu}_{N_z}$ for each $N_z$.



# Objective function

Define the average benefit after $K$ operations

$$\frac{1}{K}\sum_{k=0}^{K-1}\langle u, \frac{1}{T}\int_{T_k}^{T_{k+1}} x(t)\mathrm{d}t\rangle.$$

**Theorem (One periodic)**

*If $x$ is $KT$-periodic (i.e., $x(T_K) = x(T_0)$), then $x$ is $T-$periodic.*

$$\frac{1}{K}\sum_{k=0}^{K-1}\langle u, \frac{1}{T}\int_{T_k}^{T_{k+1}} x(t)\mathrm{d}t\rangle = \langle u, \frac{1}{T}\int_{T_0}^{T_1} x(t)\mathrm{d}t\rangle.$$