Time domain decomposition and application to PDE-constrained optimization problems

Liu-Di LU

Section of Mathematics
University of Geneva

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UNIVERSITÉ DE GENÈVE
FACULTÉ DES SCIENCES
Section de mathématiques

The **alternating Schwarz method** is the earliest domain decomposition method invented by **H.A. Schwarz** in 1870 (About a border crossing through an alternating procedure).

Ueber einen Grenzübergang durch alternirendes Verfahren.

Von

H. A. Schwarz.

(Aus einem am 30. Mai gehaltenen Vortrage.)

Die unter dem Namen Dirichlet'sches Princip bekannte Schlussweise, welche in gewissem Sinne als das Fundament des von Riemann entwickelten Zweiges der Theorie der analytischen Funktionen angesehen werden muss, unterliegt, wie jetzt wohl allgemein zugestanden wird, hinsichtlich der Strenge sehr begründeten Einwendungen, deren vollständige Entfernung, soviel ich weiss, den Anstrengungen der Mathematiker bisher nicht gelungen ist.

Historical review

Problem: Show existence of harmonic functions

$$\Delta y = 0 \ \text{ in } \Omega, \quad y = g \ \text{ on } \partial\Omega.$$

Available tools: **Fourier (1807)** for rectangular domain Ω and **Poisson (1815)** for circular domain Ω.

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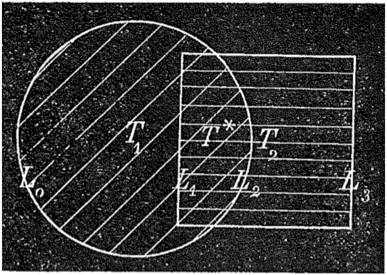
A *door handle* type domain Ω



Prove convergence of this iterative method with maximum principle.

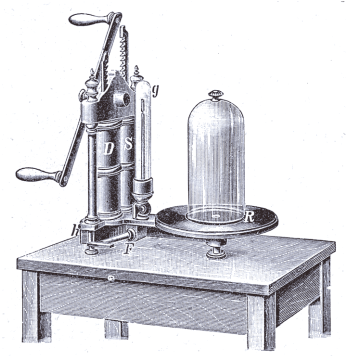
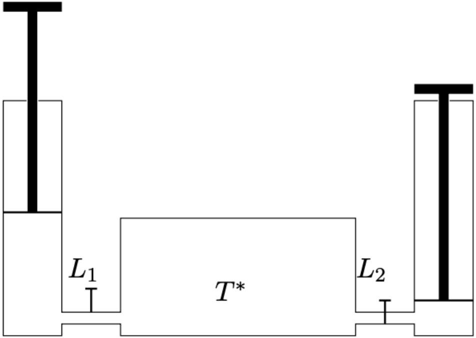
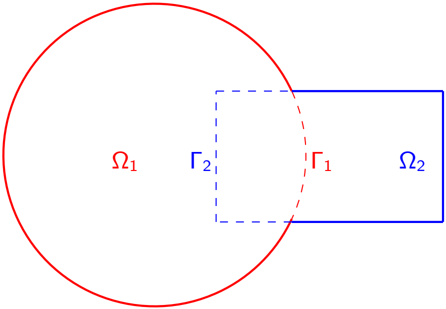


Fig. 103. Zweistiefelige Hahnluftpumpe

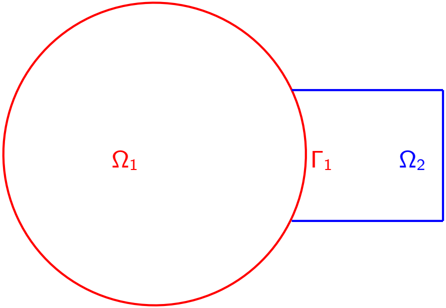


Domain: $\Omega = \Omega_1 \cup \Omega_2$



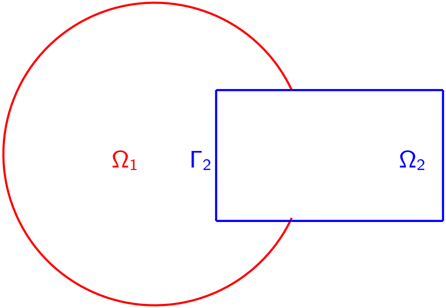
Interpretation and illustration

Domain: $\Omega = \Omega_1 \cup \Omega_2$



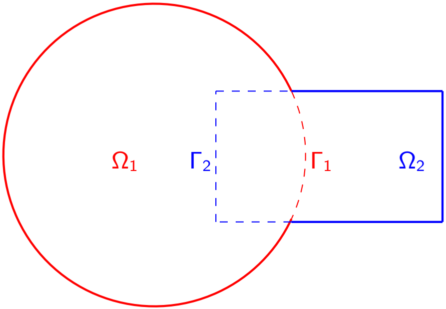
$$
\begin{aligned}
\Delta y_1^1 &= 0 && \text{in } \Omega_1, \\
y_1^1 &= g && \text{on } \partial\Omega \cap \bar{\Omega}_1, \\
y_1^1 &= y_2^0 && \text{on } \Gamma_1
\end{aligned}
$$

Domain: $\Omega = \Omega_1 \cup \Omega_2$



$$\begin{aligned} \Delta y_2^1 &= 0 && \text{in } \Omega_2, \\ y_2^1 &= g && \text{on } \partial\Omega \cap \bar{\Omega}_2, \\ y_2^1 &= y_1^1 && \text{on } \Gamma_2 \end{aligned}$$

Domain: $\Omega = \Omega_1 \cup \Omega_2$



$$\begin{aligned} \Delta y_1^\ell &= 0 && \text{in } \Omega_1, \\ y_1^\ell &= g && \text{on } \partial\Omega \cap \bar{\Omega}_1, \\ y_1^\ell &= y_2^{\ell-1} && \text{on } \Gamma_1 \end{aligned} \qquad \begin{aligned} \Delta y_2^\ell &= 0 && \text{in } \Omega_2, \\ y_2^\ell &= g && \text{on } \partial\Omega \cap \bar{\Omega}_2, \\ y_2^\ell &= y_1^\ell && \text{on } \Gamma_2 \end{aligned}$$

Towards a computational tool

- **Miller (1965)**: Numerical analogs to the Schwarz alternating procedure.
- **Dryja and Widlund (1987)**: An additive variant of the Schwarz alternating method for the case of many subregions.
- **Lions (1988,1989,1990)**: On the Schwarz alternating method I, II, III.
- **Quarteroni and Valli (1999)**: Domain decomposition methods for partial differential equations.
- **Smith, Bjorstad and Gropp (2004)**: Domain decomposition: parallel multilevel methods for elliptic partial differential equations.
- **Toselli and Widlund (2006)**: Domain decomposition methods-algorithms and theory.
- **Dolean, Jolivet and Nataf**: An Introduction to domain decomposition methods: algorithms, theory, and parallel Implementation.

Some domain decomposition methods: **alternating Schwarz**, **parallel Schwarz**, optimized Schwarz, **Dirichlet–Neumann**, Neumann–Neumann (BDD), FETI (Dirichlet–Dirichlet), ...

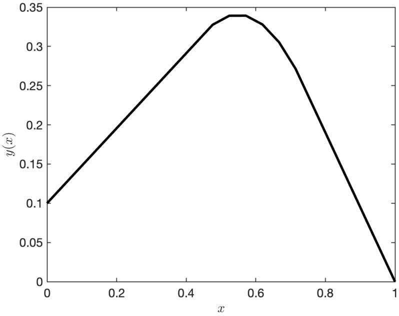
Alternating Schwarz method

Problem:

$$-\partial_{xx} y = f \quad \text{in } \Omega = (0,1),$$
$$y(0) = 0.1, \quad y(1) = 0,$$

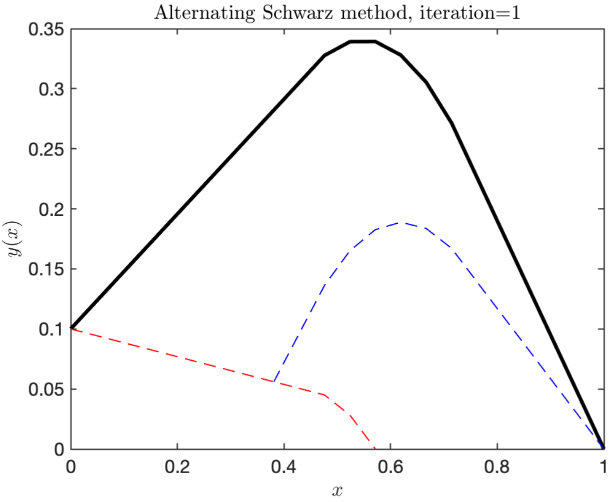
with

$$f = \begin{cases} 5 & \text{if } 0.4 < x < 0.7, \\ 0 & \text{else.} \end{cases}$$



Alternating Schwarz method

Subdomains: $\Omega_1 = (0, 0.57)$ and $\Omega_2 = (0.38, 1)$ with **an initial guess**: $y_2^0(0.57) = 0$.



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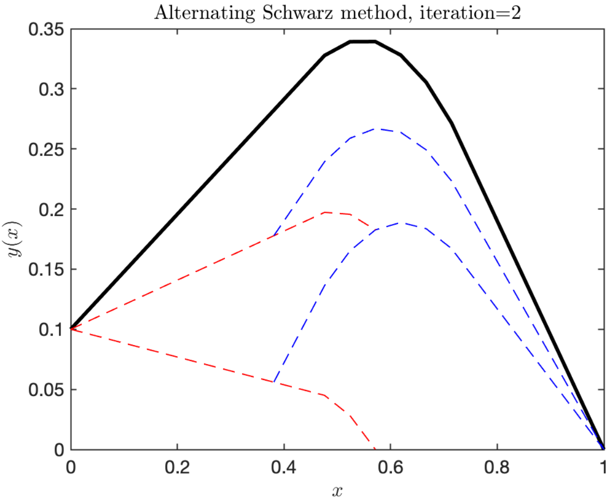

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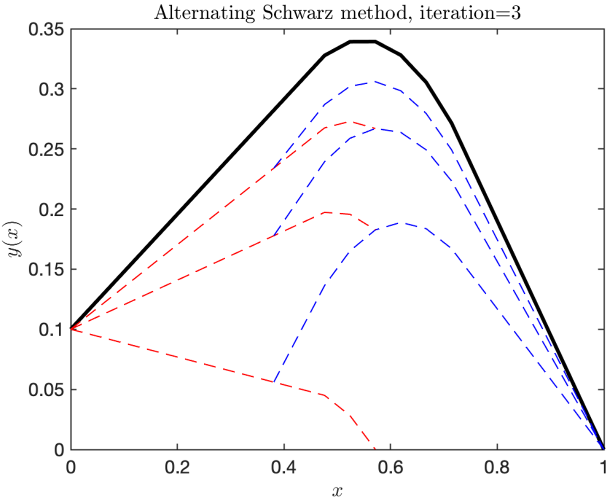

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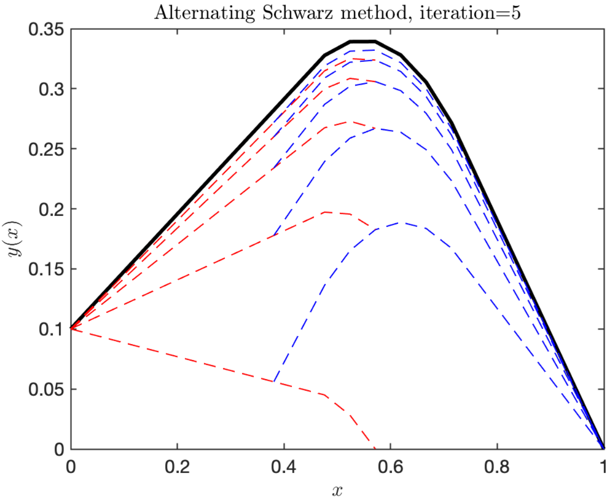

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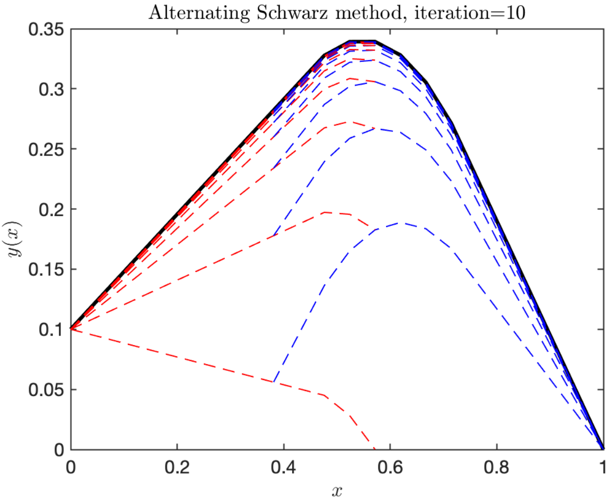

Problem:

$$-\partial_{xx} y = f \quad \text{in } \Omega = (0,1),$$
$$y(0) = g_l, \quad y(1) = g_r.$$

Subdomains: $\Omega_1 = (0, b)$ and $\Omega_2 = (a, 1)$ with $b > a$, and **an initial guess**: y_2^0, we solve

$$\begin{aligned} -\partial_{xx} y_1^\ell &= f \quad \text{in } \Omega_1, & -\partial_{xx} y_2^\ell &= f \quad \text{in } \Omega_2, \\ y_1^\ell(0) &= g_l, & y_2^\ell(1) &= g_r, \\ y_1^\ell(b) &= y_2^{\ell-1}(b), & y_2^\ell(a) &= y_1^\ell(a). \end{aligned}$$

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Denote the error $e_j^\ell := y - y_j^\ell$ that satisfies

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Convergence analysis

Denote the error $e_j^\ell := y - y_j^\ell$ that satisfies

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Analytical solutions: $e_1^\ell(x) = C_1^\ell x$ and $e_2^\ell(x) = C_2^\ell (x-1)$.

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\end{aligned}
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Analytical solutions: $e_1^\ell(x) = C_1^\ell x$ and $e_2^\ell(x) = C_2^\ell (x-1)$.
Evaluation of the coefficients C_1^ℓ and C_2^ℓ:

$$
C_1^\ell = \frac{e_2^{\ell-1}(b)}{b}, \qquad C_2^\ell = \frac{e_1^\ell(a)}{a-1} = e_2^{\ell-1}(b)\frac{1}{a-1}\frac{a}{b}.
$$

The convergence factor

$$
e_2^\ell(b) = \rho(a,b) e_2^{\ell-1}(b), \quad \rho(a,b) := \frac{1-b}{1-a}\frac{a}{b}.
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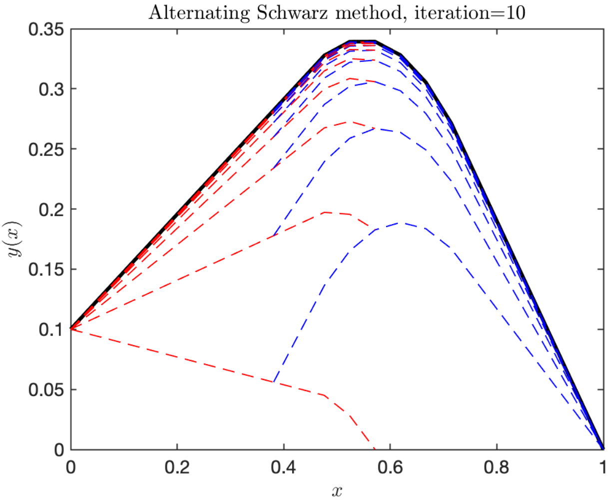
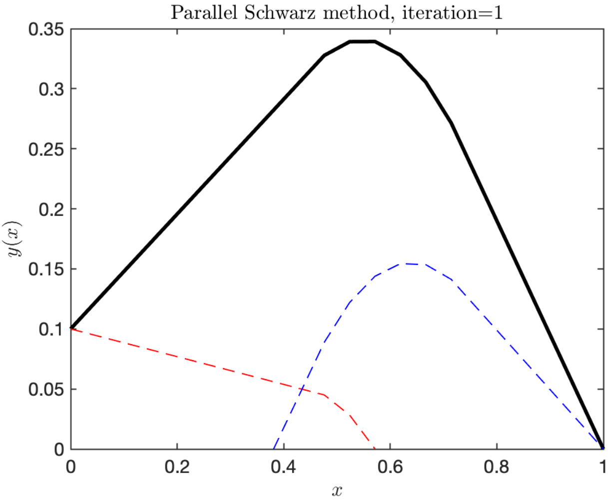
(i) The alternating Schwarz method always converges when $b > a$.
(ii) The larger the overlap size $b - a$, the better the convergence.
(iii) The alternating Schwarz method **does not converge** when $a = b$!

Parallel Schwarz method

Subdomains: $\Omega_1 = (0, 0.57)$ and $\Omega_2 = (0.38, 1)$ with **an initial guess**: $y_2^0(0.57) = 0$.



Subdomains: $\Omega_1 = (0, 0.57)$ and $\Omega_2 = (0.38, 1)$ with **two initial guesses**: $y_2^0(0.57) = 0$ and $y_1^0(0.38) = 0$.



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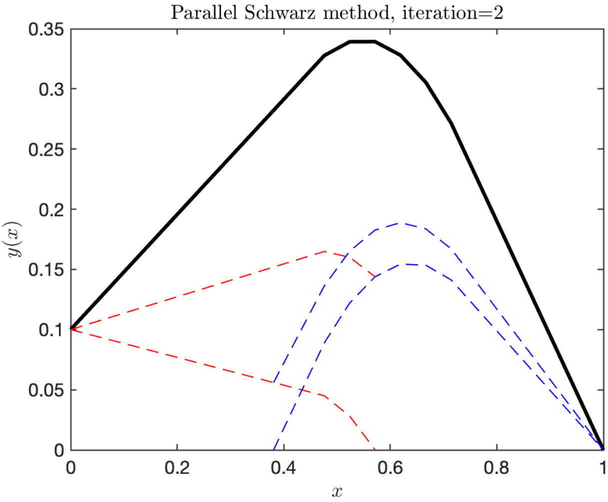

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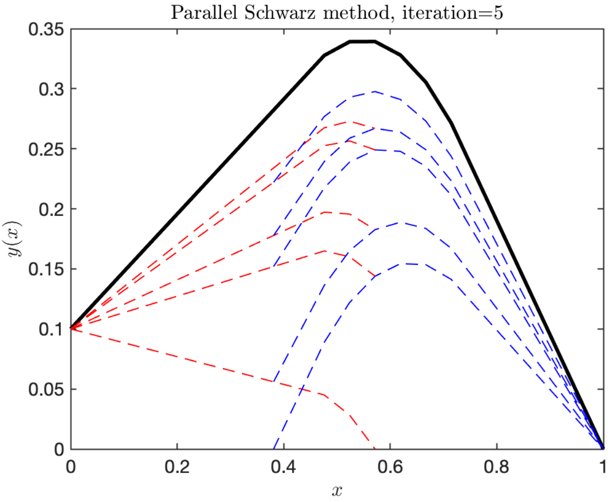

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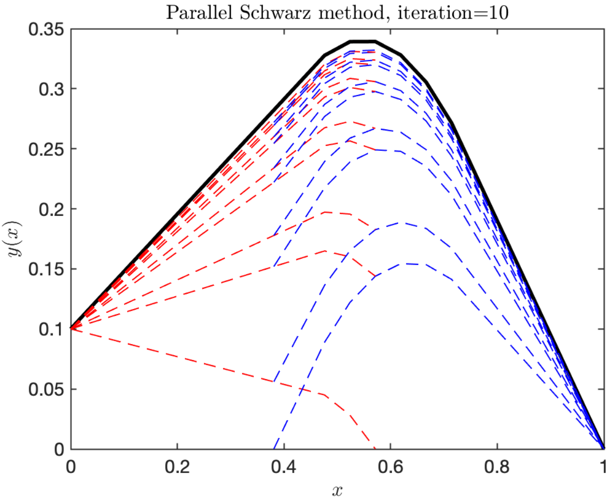

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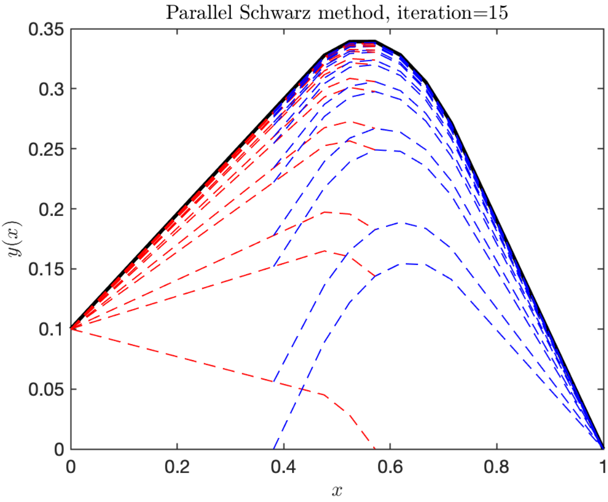

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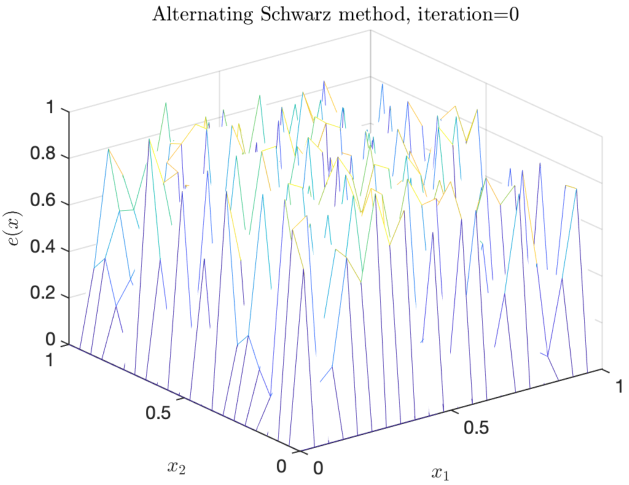
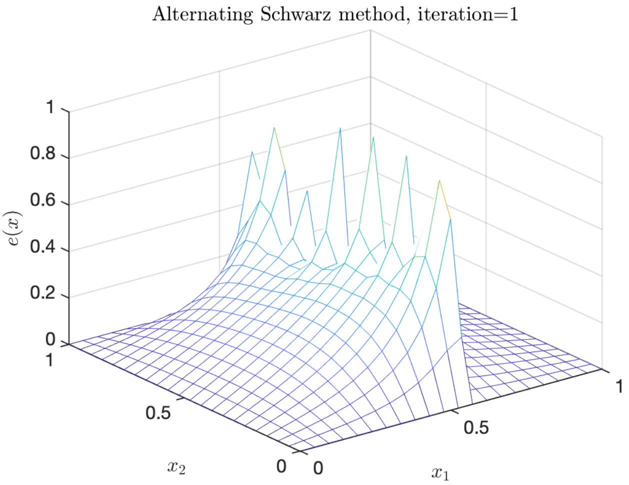
The convergence factor is now

$$e_2^{\ell+1}(b) = \frac{1-b}{1-a}\frac{a}{b}\, e_2^{\ell-1}(b).$$

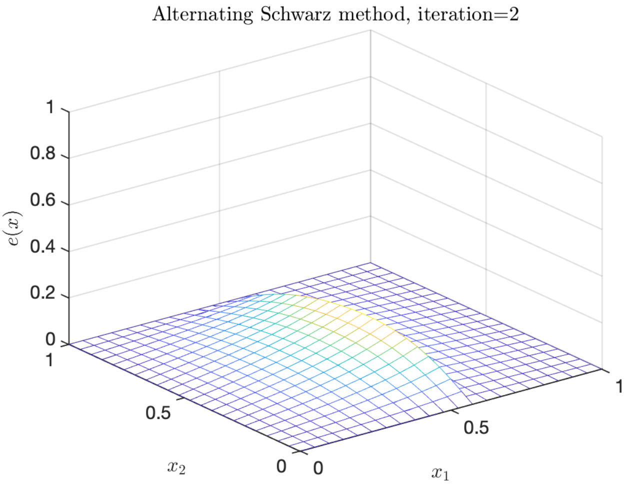
Smoother


Alternating Schwarz method, iteration=0
$e(x)$
1
0.8
0.6
0.4
0.2
0
1
0.5
0
x_2
0
0.5
1
x_1

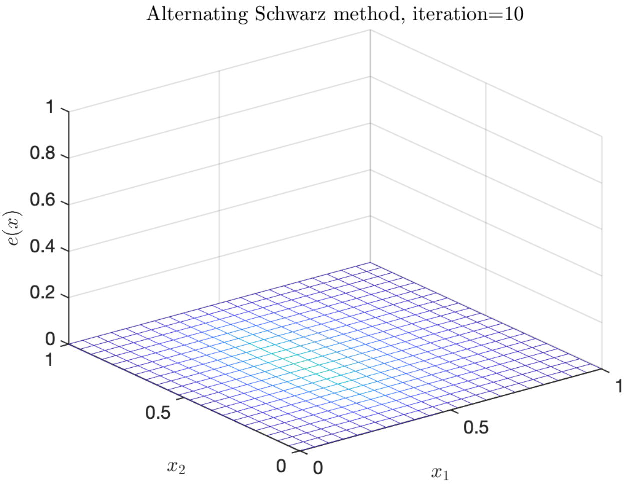
Smoother


Alternating Schwarz method, iteration=1
1
0.8
0.6
0.4
0.2
0
$e(x)$
1
0.5
0
x_2
0
0.5
1
x_1

Smoother

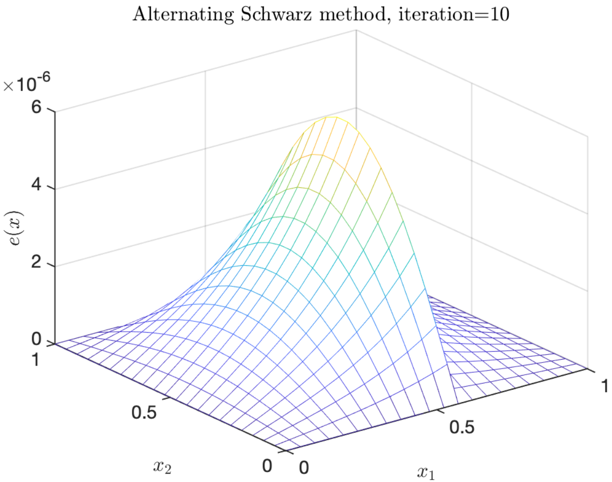

Alternating Schwarz method, iteration=2
$e(x)$
1
0.8
0.6
0.4
0.2
0
1
0.5
0
x_2
0
0.5
1
x_1

Smoother



Alternating Schwarz method, iteration=10
$e(x)$
1
0.8
0.6
0.4
0.2
0
1
0.5
0
x_2
0
0.5
1
x_1

Smoother


Alternating Schwarz method, iteration=10
×10^-6
6
4
2
0
e(x)
1
0.5
0
x2
0
0.5
1
x1

Model problem

For $\hat{y} \in L^2(Q)$, $\gamma \geq 0$ and $\nu > 0$, minimize the cost functional

$$J(y,u) := \frac{1}{2}\|y - \hat{y}\|^2_{L^2(Q)} + \frac{\gamma}{2}\|y(T) - \hat{y}(T)\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)},$$

subject to

$$\begin{aligned} \partial_t y - \Delta y &= u \quad && \text{in } Q := (0,T) \times \Omega, \\ y &= 0 && \text{on } \Sigma := (0,T) \times \partial\Omega, \\ y &= y_0 && \text{on } \Sigma_0 := \{0\} \times \Omega, \end{aligned}$$

with $\Omega \subset \mathbb{R}^n$.

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Lagrange multipliers method

$$\mathcal{L}(y,u,\lambda) = J(y,u) + \langle \partial_t y - \Delta y - u, \lambda\rangle.$$

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First-order optimality system:

$$\begin{aligned} \partial_t y - \Delta y &= u && \text{in } Q, & \partial_t\lambda + \Delta\lambda &= y - \hat{y} && \text{in } Q, \\ y &= 0 && \text{in } \Sigma, & \lambda &= 0 && \text{in } \Sigma, \\ y &= y_0 && \text{in } \Sigma_0, & \lambda &= -\gamma(y - \hat{y}) && \text{in } \Sigma_T := \{T\}\times\Omega, \end{aligned}$$

$$-\lambda + \nu u = 0 \quad \text{in } Q.$$

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Model problem

For $\hat{y} \in L^2(Q)$, $\gamma \geq 0$ and $\nu > 0$, minimize the cost functional

$$J(y,u) := \frac{1}{2}\|y-\hat{y}\|^2_{L^2(Q)} + \frac{\gamma}{2}\|y(T)-\hat{y}(T)\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)},$$

subject to

$$\begin{aligned} \partial_t y - \Delta y &= u && \text{in } Q := (0,T)\times\Omega, \\ y &= 0 && \text{on } \Sigma := (0,T)\times\partial\Omega, \\ y &= y_0 && \text{on } \Sigma_0 := \{0\}\times\Omega, \end{aligned}$$

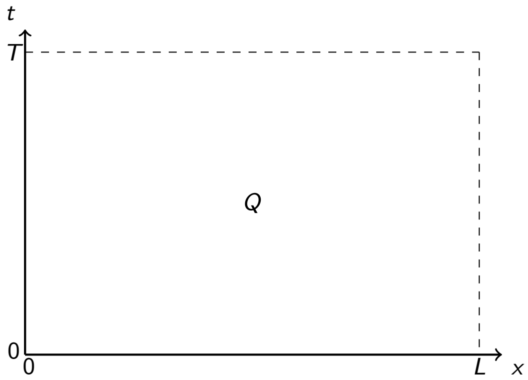
with $\Omega \subset \mathbb{R}^n$.

Lagrange multipliers method

$$\mathcal{L}(y,u,\lambda) = J(y,u) + \langle \partial_t y - \Delta y - u, \lambda\rangle.$$

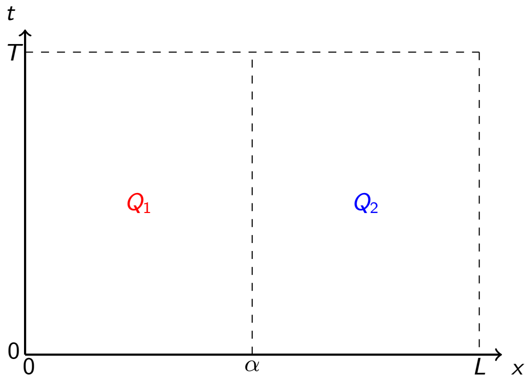
Reduced optimality system (forward-backward):

$$\begin{aligned} \partial_t y - \Delta y &= \nu^{-1}\lambda && \text{in } Q, & \partial_t\lambda + \Delta\lambda &= y - \hat{y} && \text{in } Q, \\ y &= 0 && \text{in } \Sigma, & \lambda &= 0 && \text{in } \Sigma, \\ y &= y_0 && \text{in } \Sigma_0, & \lambda &= -\gamma(y-\hat{y}) && \text{in } \Sigma_T. \end{aligned}$$



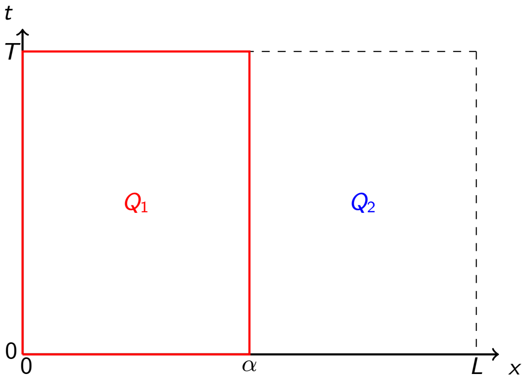
Space-time domain: $Q = (0, L) \times (0, T)$,

$$\begin{aligned} \partial_t y - \partial_{xx} y &= \nu^{-1}\lambda, & \partial_t \lambda + \partial_{xx}\lambda &= y - \hat{y}, \\ y(0,t) &= 0, & \lambda(0,t) &= 0, \\ y(L,t) &= 0, & \lambda(L,t) &= 0, \\ y(x,0) &= y_0(x), & \lambda(x,T) + \gamma y(x,T) &= \gamma \hat{y}(x,T), \end{aligned}$$



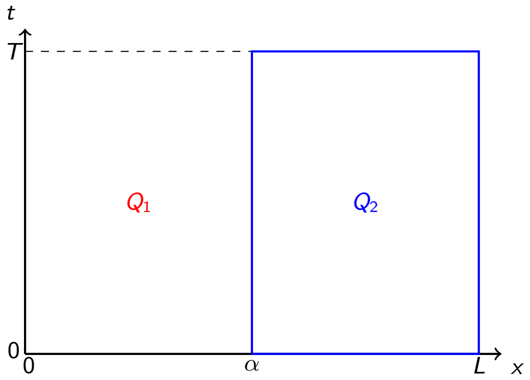
Subdomains: $Q_1 = (0, \alpha) \times (0, T)$ and $Q_2 = (\alpha, L) \times (0, T)$,

$$\begin{aligned} \partial_t y - \partial_{xx} y &= \nu^{-1}\lambda, & \partial_t \lambda + \partial_{xx}\lambda &= y - \hat{y}, \\ y(0,t) &= 0, & \lambda(0,t) &= 0, \\ y(L,t) &= 0, & \lambda(L,t) &= 0, \\ y(x,0) &= y_0(x), & \lambda(x,T) + \gamma y(x,T) &= \gamma \hat{y}(x,T), \end{aligned}$$



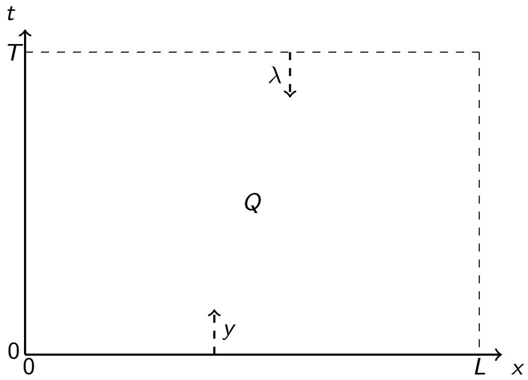
Subdomain: $Q_1 = (0,\alpha) \times (0,T)$,

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell, & \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell &= y_1^\ell - \hat{y}_1, \\ y_1^\ell(0,t) &= 0, & \lambda_1^\ell(0,t) &= 0, \\ y_1^\ell(\alpha,t) &= y_2^{\ell-1}(\alpha,t), & \lambda_1^\ell(\alpha,t) &= \lambda_2^{\ell-1}(\alpha,t), \\ y_1^\ell(x,0) &= y_{1,0}(x), & \lambda_1^\ell(x,T) + \gamma y_1^\ell(x,T) &= \gamma \hat{y}_1(x,T). \end{aligned}$$



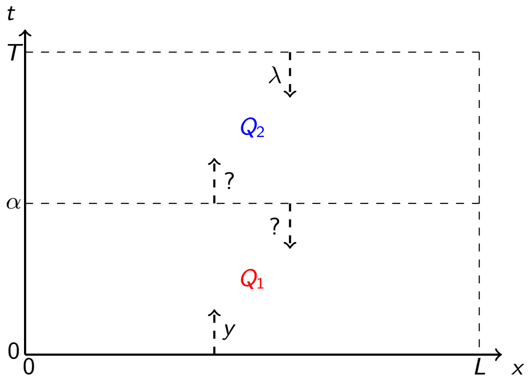
Subdomains: $Q_2 = (\alpha, 1) \times (0, T)$,

$$\begin{aligned}
\partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1}\lambda_2^\ell, & \partial_t \lambda_2^\ell + \partial_{xx}\lambda_2^\ell &= y_2^\ell - \hat{y}_2,\\
\partial_x y_2^\ell(\alpha, t) &= \partial_x y_1^\ell(\alpha, t), & \partial_x \lambda_2^\ell(\alpha, t) &= \partial_x \lambda_1^\ell(\alpha, t),\\
y_2^\ell(L, t) &= 0, & \lambda_2^\ell(L, t) &= 0,\\
y_2^\ell(x, 0) &= y_{2,0}(x), & \lambda_2^\ell(x, T) + \gamma y_2^\ell(x, T) &= \gamma \hat{y}_2(x, T).
\end{aligned}$$



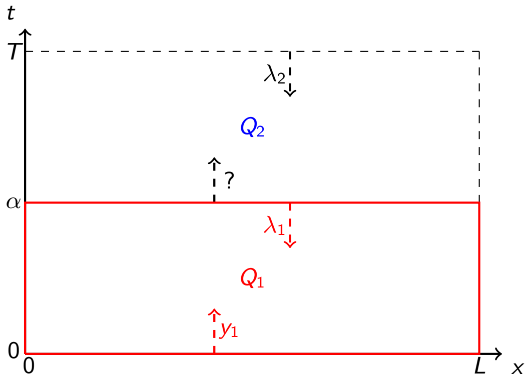
Space-time domain: $Q = (0, L) \times (0, T)$,

$$\begin{aligned} \partial_t y - \partial_{xx} y &= \nu^{-1}\lambda, & \partial_t \lambda + \partial_{xx}\lambda &= y - \hat{y}, \\ y(0,t) &= 0, & \lambda(0,t) &= 0, \\ y(L,t) &= 0, & \lambda(L,t) &= 0, \\ y(x,0) &= y_0(x), & \lambda(x,T) + \gamma y(x,T) &= \gamma \hat{y}(x,T). \end{aligned}$$



Subdomains: $Q_1 = (0, L) \times (0, \alpha)$ and $Q_2 = (0, L) \times (\alpha, T)$,

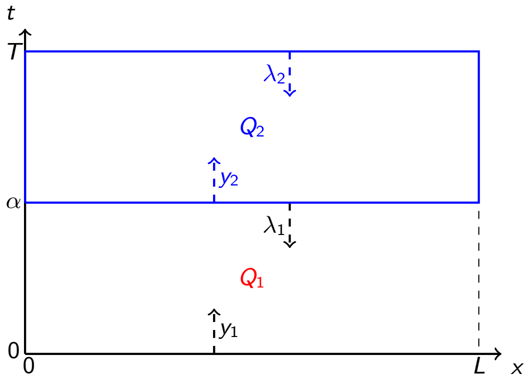
$$\begin{aligned} \partial_t y - \partial_{xx} y &= \nu^{-1}\lambda, & \partial_t \lambda + \partial_{xx}\lambda &= y - \hat{y}, \\ y(0,t) &= 0, & \lambda(0,t) &= 0, \\ y(L,t) &= 0, & \lambda(L,t) &= 0, \\ y(x,0) &= y_0(x), & \lambda(x,T) + \gamma y(x,T) &= \gamma \hat{y}(x,T). \end{aligned}$$



Subdomain: $Q_1 = (0, L) \times (0, \alpha)$,

$$\begin{aligned} \partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell, & \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell &= y_1^\ell - \hat{y}_1, \\ y_1^\ell(0,t) &= 0, & \lambda_1^\ell(0,t) &= 0, \\ y_1^\ell(L,t) &= 0, & \lambda_1^\ell(L,t) &= 0, \\ y_1^\ell(x,0) &= y_0(x), & \lambda_1^\ell(x,\alpha) &= \lambda_2^{\ell-1}(x,\alpha), \end{aligned}$$

Time domain decomposition



Subdomain: $Q_2 = (0, L) \times (\alpha, T)$,

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1}\lambda_2^\ell, & \partial_t \lambda_2^\ell + \partial_{xx}\lambda_2^\ell &= y_2^\ell - \hat{y}_2, \\ y_2^\ell(0,t) &= 0, & \lambda_2^\ell(0,t) &= 0, \\ y_2^\ell(L,t) &= 0, & \lambda_2^\ell(L,t) &= 0, \\ \partial_t y_2^\ell(x,\alpha) &= \partial_t y_1^\ell(x,\alpha), & \lambda_2^\ell(x,T) + \gamma y_2^\ell(x,T) &= \gamma \hat{y}_2(x,T). \end{aligned}$$

Reduced optimality system:

$$\begin{cases} \partial_t \begin{pmatrix} y \\ \lambda \end{pmatrix} + \begin{pmatrix} -\partial_{xx} y - \nu^{-1}\lambda \\ -y + \partial_{xx}\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{y} \end{pmatrix}, \\ \qquad y(\cdot, 0) = y_0(\cdot), \\ \qquad \lambda(\cdot, T) + \gamma y(\cdot, T) = \gamma \hat{y}(\cdot, T), \end{cases} \quad \text{in } (0, L) \times (0, T)$$

Semi-discretization

Finite difference discretization: $-\partial_{xx} \approx A \in \mathbb{R}^{n\times n}$,

$$\begin{cases} \begin{pmatrix} \dot{\boldsymbol{y}} \\ \dot{\boldsymbol{\lambda}} \end{pmatrix} + \begin{pmatrix} A & -\nu^{-1} I \\ -I & -A \end{pmatrix} \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{\boldsymbol{y}} \end{pmatrix} \text{ in } (0, T), \\ \boldsymbol{y}(0) = \boldsymbol{y}_0, \\ \boldsymbol{\lambda}(T) + \gamma \boldsymbol{y}(T) = \gamma \hat{\boldsymbol{y}}(T), \end{cases}$$

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Diagonalization: $A = PDP^{-1}$ and $D = \operatorname{diag}(d_1, \dots, d_n)$,

$$\begin{cases} \begin{pmatrix} \dot{z}_i \\ \dot{\mu}_i \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_i \\ \mu_i \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{z}_i \end{pmatrix} \text{ in } (0,T), \\ z_i(0) = z_{0,i}, \\ \mu_i(T) + \gamma z_i(T) = \gamma \hat{z}_i(T), \end{cases}$$

with $\boldsymbol{z} = P^{-1}\boldsymbol{y}$, $\hat{\boldsymbol{z}} = P^{-1}\hat{\boldsymbol{y}}$ and $\boldsymbol{\mu} = P^{-1}\boldsymbol{\lambda}$. So n **independent** 2×2 **systems**.

Semi-discretization

Finite difference discretization: $-\partial_{xx} \approx A \in \mathbb{R}^{n\times n}$,

$$\begin{cases} \begin{pmatrix} \dot{\boldsymbol{y}} \\ \dot{\boldsymbol{\lambda}} \end{pmatrix} + \begin{pmatrix} A & -\nu^{-1} I \\ -I & -A \end{pmatrix} \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{\boldsymbol{y}} \end{pmatrix} \text{ in } (0,T), \\ \boldsymbol{y}(0) = \boldsymbol{y}_0, \\ \boldsymbol{\lambda}(T) + \gamma \boldsymbol{y}(T) = \gamma \hat{\boldsymbol{y}}(T), \end{cases}$$

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with $\boldsymbol{z} = P^{-1}\boldsymbol{y}$, $\hat{\boldsymbol{z}} = P^{-1}\hat{\boldsymbol{y}}$ and $\boldsymbol{\mu} = P^{-1}\boldsymbol{\lambda}$. So n **independent** 2×2 **systems**.
Second-order ODE:

$$\begin{cases} \ddot{z}_i - \sigma_i^2 z_i = -\nu^{-1}\hat{z}_i \text{ in } (0,T), \\ z_i(0) = z_{0,i}, \\ \dot{z}_i(T) + \omega_i z_i(T) = \nu^{-1}\gamma \hat{z}_i(T), \end{cases}$$

with $\sigma_i := \sqrt{d_i^2 + \nu^{-1}}$, $\omega_i := \nu^{-1}\gamma + d_i$ and $\beta_i := 1 - \gamma d_i$.

Semi-discretization

Finite difference discretization: $-\partial_{xx} \approx A \in \mathbb{R}^{n\times n}$,

$$\begin{cases} \begin{pmatrix} \dot{\boldsymbol{y}} \\ \dot{\boldsymbol{\lambda}} \end{pmatrix} + \begin{pmatrix} A & -\nu^{-1} I \\ -I & -A \end{pmatrix} \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{\boldsymbol{y}} \end{pmatrix} \text{ in } (0, T), \\ \boldsymbol{y}(0) = \boldsymbol{y}_0, \\ \boldsymbol{\lambda}(T) + \gamma \boldsymbol{y}(T) = \gamma \hat{\boldsymbol{y}}(T), \end{cases}$$

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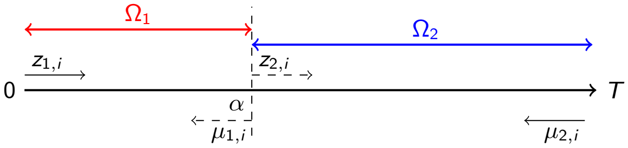
with $\boldsymbol{z} = P^{-1}\boldsymbol{y}$, $\hat{\boldsymbol{z}} = P^{-1}\hat{\boldsymbol{y}}$ and $\boldsymbol{\mu} = P^{-1}\boldsymbol{\lambda}$. So n **independent** 2×2 **systems**.
Second-order ODE:

$$\begin{cases} \ddot{\mu}_i - \sigma_i^2 \mu_i = -(\dot{\hat{z}}_i + d_i \hat{z}_i) \text{ in } (0, T), \\ \mu_i(0) - d_i \mu_i(0) = z_{0,i} - \hat{z}_i(0), \\ \gamma \dot{\mu}_i(T) + \beta_i \mu_i(T) = 0, \end{cases}$$

with $\sigma_i := \sqrt{d_i^2 + \nu^{-1}}$, $\omega_i := \nu^{-1}\gamma + d_i$ and $\beta_i := 1 - \gamma d_i$.

Transformation



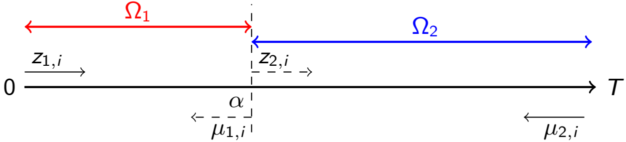
Dirichlet:

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^{\ell} \\ \dot{\mu}_{1,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^{\ell} \\ \mu_{1,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ \mu_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases}$$

Update:

$$f_{\alpha,i}^{\ell} := (1-\theta) f_{\alpha,i}^{\ell-1} + \theta \mu_{2,i}^{\ell}(\alpha), \quad \theta \in (0,1).$$

Transformation



Dirichlet:

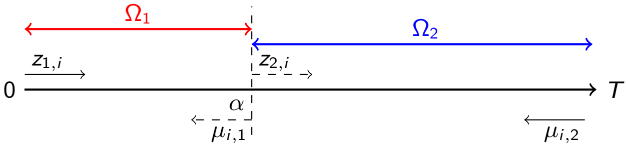
$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^{\ell} \\ \dot{\mu}_{1,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^{\ell} \\ \mu_{1,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ \mu_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases}$$

Neumann:

$$\begin{cases} \begin{pmatrix} \dot{z}_{2,i}^{\ell} \\ \dot{\mu}_{2,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^{\ell} \\ \mu_{2,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}_{2,i}^{\ell}(\alpha) = \dot{z}_{1,i}^{\ell}(\alpha), \\ \mu_{2,i}^{\ell}(T) + \gamma z_{2,i}^{\ell}(T) = 0, \end{cases}$$

Update:

$$f_{\alpha,i}^{\ell} := (1-\theta) f_{\alpha,i}^{\ell-1} + \theta \mu_{2,i}^{\ell}(\alpha), \quad \theta \in (0,1).$$



Dirichlet:

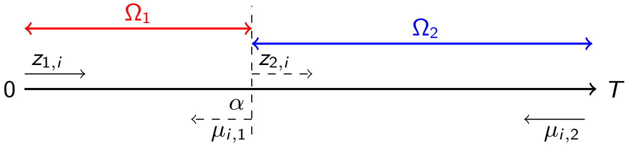
$$\begin{cases} \ddot{z}_{1,i}^{\ell} - \sigma_i^2 z_{1,i}^{\ell} = 0 \text{ in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ \dot{z}_{1,i}^{\ell}(\alpha) + d_i z_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases}$$

Neumann:

$$\begin{cases} \ddot{z}_{2,i}^{\ell} - \sigma_i^2 z_{2,i}^{\ell} = 0 \text{ in } \Omega_2, \\ \dot{z}_{2,i}^{\ell}(\alpha) = \dot{z}_{1,i}^{\ell}(\alpha), \\ \dot{z}_{2,i}^{\ell}(T) + \omega_i z_{2,i}^{\ell}(T) = 0, \end{cases}$$

Update:

$$f_{\alpha,i}^{\ell} = (1-\theta) f_{\alpha,i}^{\ell-1} + \theta\big(\dot{z}_{2,i}^{\ell}(\alpha) + d_i z_{2,i}^{\ell}(\alpha)\big).$$



Dirichlet:

$$\begin{cases} \ddot{\mu}_{1,i}^{\ell} - \sigma_i^2 \mu_{1,i}^{\ell} = 0 \text{ in } \Omega_1, \\ \dot{\mu}_i(0) - d_i \mu_i(0) = 0, \\ \mu_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases}$$

Neumann:

$$\begin{cases} \ddot{\mu}_{2,i}^{\ell} - \sigma_i^2 \mu_{2,i}^{\ell} = 0 \text{ in } \Omega_2, \\ \ddot{\mu}_{2,i}^{\ell}(\alpha) - d_i \dot{\mu}_{2,i}^{\ell}(\alpha) = \ddot{\mu}_{1,i}^{\ell}(\alpha) - d_i \dot{\mu}_{1,i}^{\ell}(\alpha), \\ \gamma \dot{\mu}_i^{\ell}(T) + \beta_i \mu_i^{\ell}(T) = 0, \end{cases}$$

Update:

$$f_{\alpha,i}^{\ell} = (1-\theta) f_{\alpha,i}^{\ell-1} + \theta \mu_{2,i}^{\ell}(\alpha), \quad \theta \in (0,1).$$

Transformation

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^{\ell} \\ \dot{\mu}_{1,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^{\ell} \\ \mu_{1,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ \mu_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases}$$

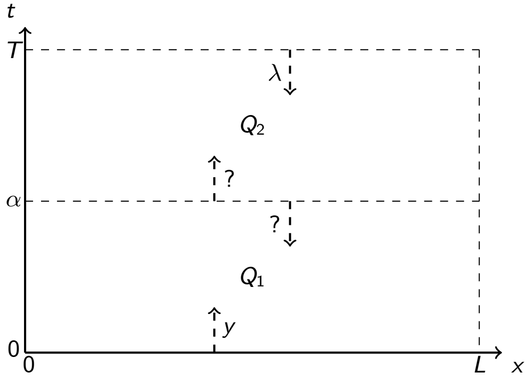
$$\begin{cases} \begin{pmatrix} \dot{z}_{2,i}^{\ell} \\ \dot{\mu}_{2,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^{\ell} \\ \mu_{2,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}_{2,i}^{\ell}(\alpha) = \dot{z}_{1,i}^{\ell}(\alpha), \\ \mu_{2,i}^{\ell}(T) + \gamma z_{2,i}^{\ell}(T) = 0, \end{cases}$$

$$f_{\alpha,i}^{\ell} := (1-\theta) f_{\alpha,i}^{\ell-1} + \theta \mu_{2,i}^{\ell}(\alpha), \quad \theta \in (0,1).$$

Two observations:

(1) Three systems are equivalent, so same convergence using z or μ;

(2) Not anymore a "DN" algorithm, and forward-backward structure is less important.



Space-time domain: $Q = (0, L) \times (0, T)$,

$$
\begin{aligned}
\partial_t y - \Delta y &= \nu^{-1}\lambda, & \partial_t \lambda + \Delta \lambda &= y - \hat{y},\\
y(0,t) &= 0, & \lambda(0,t) &= 0,\\
y(L,t) &= 0, & \lambda(L,t) &= 0,\\
y(x,0) &= y_0(x), & \lambda(x,T) + \gamma y(x,T) &= \gamma \hat{y}(x,T).
\end{aligned}
$$

Variants of DN and ND algorithms

Category	Ω_1	Ω_2	type
(z_i, μ_i)	μ_i	$\dot{z}_i$	(DN)
	$\dot{z}_i + d_i z_i$	$\dot{z}_i$	(RN)
	$\dot{\mu}_i$	z_i	(ND)
	$\ddot{z}_i + d_i \dot{z}_i$	z_i	(RD)
z_i	z_i	$\dot{z}_i$	(DN)
	z_i	$\dot{z}_i$	(DN)
	$\dot{z}_i$	z_i	(ND)
	$\dot{z}_i$	z_i	(ND)
μ_i	μ_i	$\dot{\mu}_i$	(DN)
	$\dot{z}_i + d_i z_i$	$\ddot{z}_i + d_i \dot{z}_i$	(RR)
	$\dot{\mu}_i$	μ_i	(ND)
	$\ddot{z}_i + d_i \dot{z}_i$	$\dot{z}_i + d_i z_i$	(RR)

Comparaison of two DN variants

Natural Dirichlet–Neumann (DN_1):

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^{\ell} \\ \dot{\mu}_{1,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^{\ell} \\ \mu_{1,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ \mu_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases} \quad \begin{cases} \begin{pmatrix} \dot{z}_{2,i}^{\ell} \\ \dot{\mu}_{2,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^{\ell} \\ \mu_{2,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}_{2,i}^{\ell}(\alpha) = \dot{z}_{1,i}^{\ell}(\alpha), \\ \mu_{2,i}^{\ell}(T) + \gamma z_{2,i}^{\ell}(T) = 0, \end{cases}$$

$$f_{\alpha,i}^{\ell} := (1-\theta) f_{\alpha,i}^{\ell-1} + \theta \mu_{2,i}^{\ell}(\alpha).$$

Dirichlet–Neumann at two levels (DN_2):

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^{\ell} \\ \dot{\mu}_{1,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^{\ell} \\ \mu_{1,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ z_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases} \quad \begin{cases} \begin{pmatrix} \dot{z}_{2,i}^{\ell} \\ \dot{\mu}_{2,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^{\ell} \\ \mu_{2,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}_{2,i}^{\ell}(\alpha) = \dot{z}_{1,i}^{\ell}(\alpha), \\ \mu_{2,i}^{\ell}(T) + \gamma z_{2,i}^{\ell}(T) = 0, \end{cases}$$

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Forward–backward structure can always be recovered !

$$z_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1} \Rightarrow \dot{\mu}_{1,i}^{\ell}(\alpha) - d_i \mu_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}.$$

Comparaison of two DN variants

Natural Dirichlet–Neumann (DN_1):

$$\begin{cases} \ddot{z}_{1,i}^{\ell} - \sigma_i^2 z_{1,i}^{\ell} = 0 \text{ in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ \dot{z}_{1,i}^{\ell}(\alpha) + d_i z_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases} \qquad \begin{cases} \ddot{z}_{2,i}^{\ell} - \sigma_i^2 z_{2,i}^{\ell} = 0 \text{ in } \Omega_2, \\ \dot{z}_{2,i}^{\ell}(\alpha) = \dot{z}_{1,i}^{\ell}(\alpha), \\ \dot{z}_{2,i}^{\ell}(T) + \omega_i z_{2,i}^{\ell}(T) = 0, \end{cases}$$

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Solve the problem and find

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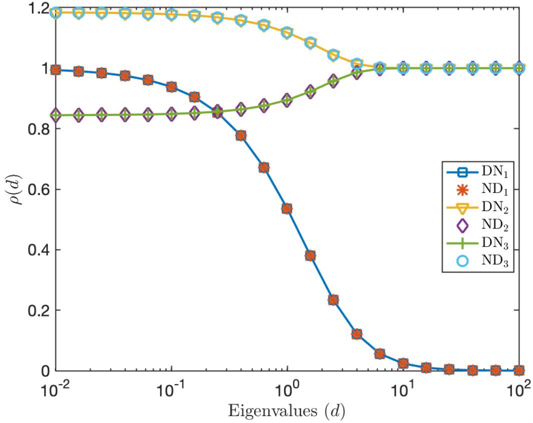
"Optimal" relaxation parameter with equioscillation

$$\theta^*_{\mathrm{DN}_2} = \frac{2}{3 + \coth(\sqrt{\nu^{-1}}\alpha)\frac{\coth\left(\sqrt{\nu^{-1}}(T-\alpha)\right) + \gamma\sqrt{\nu^{-1}}}{1 + \gamma\sqrt{\nu^{-1}}\coth\left(\sqrt{\nu^{-1}}(T-\alpha)\right)}},$$

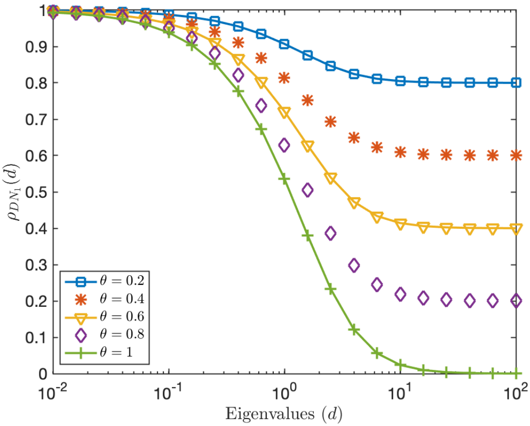
$$\theta^*_{\mathrm{ND}_2} = \frac{2}{3 + \tanh(\sqrt{\nu^{-1}}\alpha)\frac{\tanh\left(\sqrt{\nu^{-1}}(T-\alpha)\right) + \gamma\sqrt{\nu^{-1}}}{1 + \gamma\sqrt{\nu^{-1}}\tanh\left(\sqrt{\nu^{-1}}(T-\alpha)\right)}},$$

$\theta^*_{\mathrm{DN}_2} = \theta^*_{\mathrm{ND}_3}$ and $\theta^*_{\mathrm{ND}_2} = \theta^*_{\mathrm{DN}_3}$.

Convergence factor of different DN and ND variants, with penalization parameters: $\nu = 0.1$, $\gamma = 0$, interface: $\alpha = \frac{T}{2}$, and relaxation parameter: $\theta = 1$.

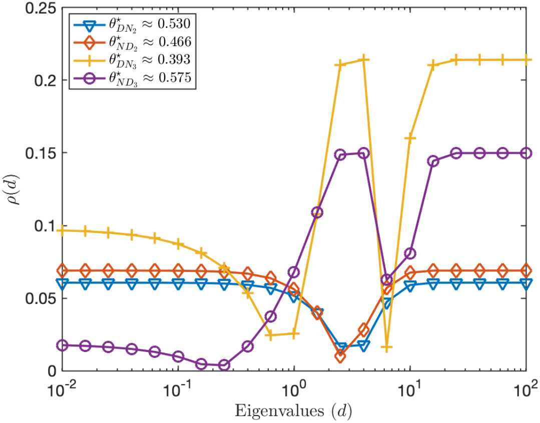


Convergence factor of different DN_1, with penalization parameters: $\nu = 0.1$, $\gamma = 0$, interface: $\alpha = \frac{T}{2}$.



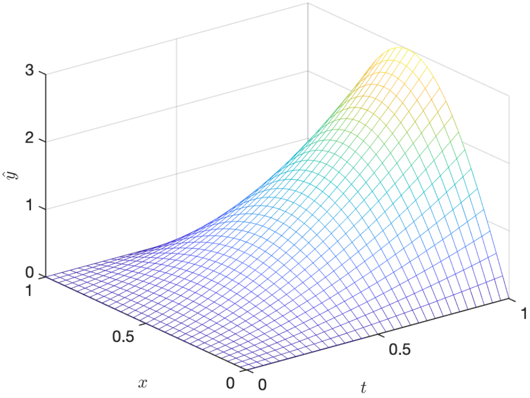
Numerical experiments

Optimal convergence factors with penalization parameters: $\nu = 0.1$, $\gamma = 10$ and interface: $\alpha = \frac{7}{10}T$.



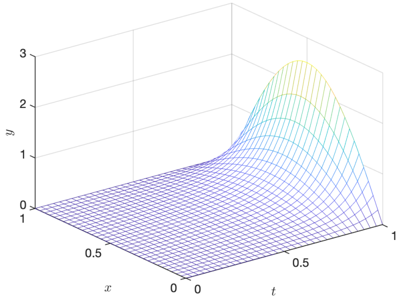
$\theta^{\star}_{\mathrm{DN}_2} = \theta^{*}_{\mathrm{DN}_2}$ and $\theta^{\star}_{\mathrm{ND}_2} = \theta^{*}_{\mathrm{ND}_2}$.

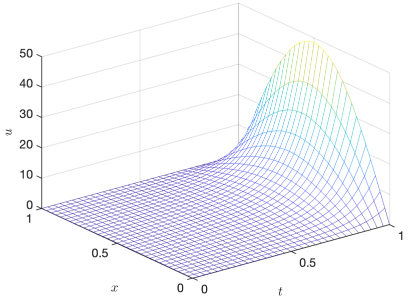
Numerical tests with penalization parameters: $\nu = 0.1$, $\gamma = 10$, final time: $T = 1$, and a target function $\hat{y}(t, x) = \sin(\pi x)(2t^2 + t)$.



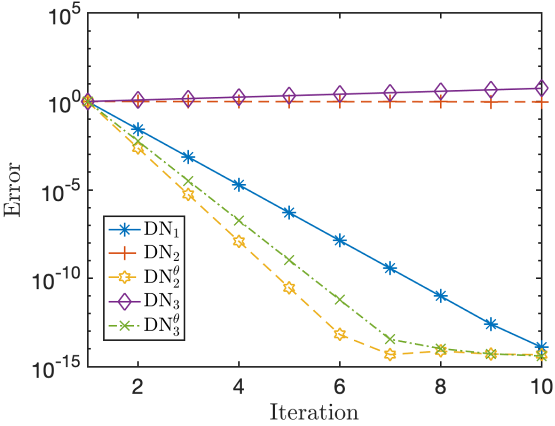
Numerical experiments

Numerical scheme: Crank-Nicolson, and mesh size: $h_t = h_x = \frac{1}{32}$.





With an interface $\alpha = \frac{7}{10}T$.



Conclusion

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- Like always, there are a lot of interesting things to be further discovered ;)

Thanks for your attention !