Some optimization problems in an algal raceway pond

Olivier Bernard, Liu-Di LU, Jacques Sainte-Marie, Julien Salomon

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Introduction

- Motivation: High potential on commercial applications, e.g., cosmetics, pharmaceuticals, food complements, wastewater treatment, green energy, etc.
- Raceway ponds

Figure: A typical raceway for cultivating microalgae. Notice the paddle-wheel which mixes the culture suspension. Picture from INRA (ANR Symbiose project) [1].

1D Illustration

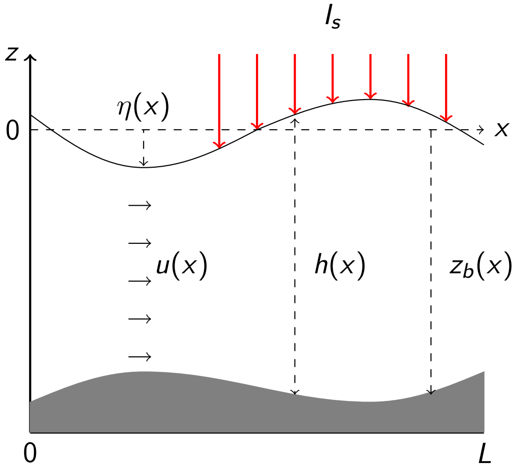

Figure: Representation of the hydrodynamic model.

Saint-Venant Equations

- 1D steady state Saint-Venant equations

$$\partial_x(hu) = 0, \tag{1}$$

$$\partial_x(hu^2 + g\frac{h^2}{2}) = -gh\partial_x z_b. \tag{2}$$

Saint-Venant Equations

- u, z_b as a function of h

$$u = \frac{Q_0}{h}, \tag{1}$$

$$z_b = \frac{M_0}{g} - \frac{Q_0^2}{2gh^2} - h, \tag{2}$$

  $Q_0, M_0 \in \mathbb{R}^+$ are two constants.

- Froude number:

$$Fr := \frac{u}{\sqrt{gh}}$$

  $Fr < 1$: subcritical case (i.e. the flow regime is fluvial)
  $Fr > 1$: supercritical case (i.e. the flow regime is torrential)

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- Given a smooth topography z_b, there exists a unique positive smooth solution of h which satisfies the subcritical flow condition [4, Lemma 1].

Lagrangian Trajectories

- Incompressibility of the flow: $\nabla \cdot \underline{\mathbf{u}} = 0$ with $\underline{\mathbf{u}} = (u(x), w(x,z))$

$$\partial_x u + \partial_z w = 0. \tag{3}$$

- Integrating (3) from z_b to z and using the kinematic condition at bottom $(w(x, z_b) = u(x)\partial_x z_b)$ gives:

$$w(x,z) = (\frac{M_0}{g} - \frac{3u^2(x)}{2g} - z)u'(x).$$

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- The Lagrangian trajectory is characterized by the system

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- A time free formulation of the Lagrangian trajectory:

$$z(x) = \eta(x) + \frac{h(x)}{h(0)}(z(0) - \eta(0)). \tag{4}$$

Han model and connection

- Reduced Han model:

$$\dot{C} = -(k_d \tau \frac{(\sigma I)^2}{\tau \sigma I + 1} + k_r) C + k_d \tau \frac{(\sigma I)^2}{\tau \sigma I + 1}.$$

- The net growth rate:

$$\mu(C, I) := k \sigma I A - R = k \sigma I \frac{(1 - C)}{\tau \sigma I + 1} - R,$$

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- The Beer-Lambert law describes how light is attenuated with depth

$$I(x, z) = I_s \exp\Big(-\varepsilon(\eta(x) - z)\Big), \tag{5}$$

where ε is the light extinction defined by:

$$\varepsilon = \frac{1}{h}\ln(\frac{I_s}{I_b}).$$

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$$\bar{\mu}_\infty := \frac{1}{V}\int_0^L \int_{z_b(x)}^{\eta(x)} \mu\big(C(x,z), I(x,z)\big)\mathrm{d}z\mathrm{d}x,$$

$$\bar{\mu}_{N_z} := \frac{1}{VN_z}\sum_{i=1}^{N_z}\int_0^L \mu(C_i, I_i)h\mathrm{d}x.$$

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- Volume of the system

$$V = \int_0^L h(x)\mathrm{d}x. \tag{6}$$

- Parameterize h by a vector $a := [a_1, \cdots, a_N] \in \mathbb{R}^N$.

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- Parameterize h by a vector $a := [a_1, \cdots, a_N] \in \mathbb{R}^N$.
- The computational chain:

$$a \to h \to z_i \to I_i \to C_i \to \bar{\mu}_{N_z}.$$

- Optimization Problem: $\bar{\mu}_{N_z}(a) = \frac{1}{VN_z}\sum_{i=1}^{N_z}\int_0^L \mu(C_i, I_i(a))h(a)\mathrm{d}x$, where C_i satisfy

$$C_i' = \left(-\alpha\left(I_i(a)\right)C_i + \beta\left(I_i(a)\right)\right)\frac{h(a)}{Q_0}.$$

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- Lagrangian

$$\begin{aligned}\mathcal{L}(C_i, a, p_i) = \frac{1}{VN_z}\sum_{i=1}^{N_z}\int_0^L \Big(-\gamma(I_i(a))C_i + \zeta(I_i(a))\Big)h(a)\mathrm{d}x \\ -\sum_{i=1}^{N_z}\int_0^L p_i\Big(C_i' + \frac{\alpha(I_i(a)) - \beta(I_i(a))}{Q_0}h(a)\Big)\mathrm{d}x.\end{aligned}$$

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- The gradient $\nabla\bar{\mu}_{N_z}(a) = \partial_a\mathcal{L}$ is given by

$$\partial_a\mathcal{L} = \sum_{i=1}^{N_z}\int_0^L \Big(\frac{-\gamma'\left(I_i\right)C_i + \zeta'\left(I_i\right)}{VN_z} + p_i\frac{-\alpha'(I_i)C_i + \beta'(I_i)}{Q_0}\Big)h\partial_a I_i\mathrm{d}x \\ +\sum_{i=1}^{N_z}\int_0^L \Big(\frac{-\gamma\left(I_i\right)C_i + \zeta\left(I_i\right)}{VN_z} + p_i\frac{-\alpha(I_i)C_i + \beta(I_i)}{Q_0}\Big)\partial_a h\mathrm{d}x.$$

Numerical settings

Parameterization of h: Truncated Fourier

$$h(x) = a_0 + \sum_{n=1}^{N} a_n \sin(2n\pi \frac{x}{L}). \tag{7}$$

Parameter to be optimized: Fourier coefficients $a := [a_1, \ldots, a_N]$. We use this parameterization based on the following reasons :

- We consider a hydrodynamic regime where the solutions of the shallow water equations are smooth and hence the water depth can be approximated by (7).
- One has naturally $h(0) = h(L)$ under this parameterization, which means that we have accomplished one lap of the raceway pond.
- We assume a constant volume of the system V, which can be achieved by fixing a_0. Indeed, under this parameterization and using (6), one finds $V = a_0 L$.

Convergence

We fix $N = 5$ and take 100 random initial guesses of a. For N_z varying from 1 to 80, we compute the average value of $\bar{\mu}_{N_z}$ for each N_z.

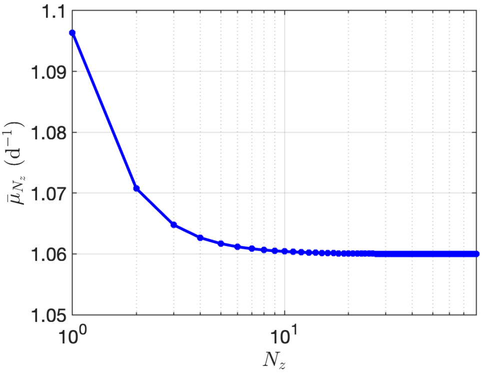
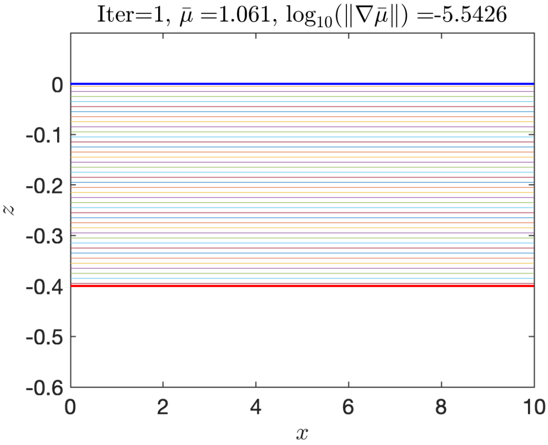


Figure: The value of $\bar{\mu}_{N_z}$ for $N_z = [1, 80]$.

Optimal Topography

We take $N_z = 40$. As an initial guess, we consider the flat topography, meaning that a is set to 0.



Periodic case

Assumption

Photoinhibition state C is periodic meaning that $C_i(L) = C_i(0)$

Consequence

Differentiating $\mathcal{L}$ with respect to $C_i(L)$, we have

$$\partial_{C_i(L)}\mathcal{L} = p_i(L) - p_i(0).$$

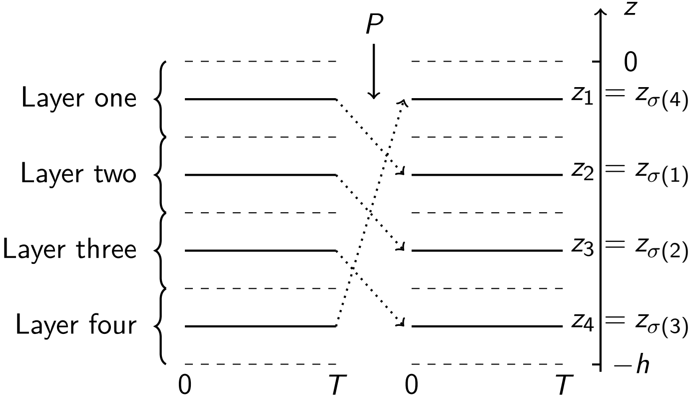
so that equating the above equation to zero gives the periodicity for p_i.

Theorem (Flat topography [2])

Assume the volume of the system V is constant. Then $\nabla \bar{\mu}_{N_z}(0) = 0$.

Mixing devices

- An ideal rearrangement of trajectories: at each new lap, the algae at depth $z_i(0)$ are entirely transferred into the position $z_j(0)$ when passing through the mixing device.
- We denote by $\mathcal{P}$ the set of permutation matrices of size $N \times N$ and by $\mathfrak{S}_N$ the associated set of permutations of N elements.



General problem

Given a period T, and initial time T_0 and a sequence $(T_k)_{k\in\mathbb{N}}$, with $T_k = kT + T_0$, we consider the following resource allocation problem:

Periodic dynamical resource allocation problem

Consider N resources denoted by $(I_n)_{n=1}^N \in \mathbb{R}^N$ which can be allocated to N activities denoted by $(x_n)_{n=1}^N$ where x_n consists of a real function of time. On a time interval $[T_k, T_{k+1})$, each activity uses the assigned resource and evolves according to a linear dynamics

$$\dot{x}_n = -\alpha(I_n)x_n + \beta(I_n), \tag{8}$$

where $\alpha : \mathbb{R} \to \mathbb{R}_+$ and $\beta : \mathbb{R} \to \mathbb{R}_+$ are given. At time T_{k+1}, the resources is re-assigned, meaning that $x(T_{k+1}) = Px(T_{k+1}^-)$ for some $P \in \mathcal{P}$. In this way, $k \in \mathbb{N}$ represents the number of re-assignments and T_k^- represents the moment just before re-assignment.

Assumption

Resource $(I_n)_{n=1}^N$ are constant with respect to time.

Consequence

For a given initial vector of states $(x_n(T_0))_{n=1}^N$, we have

$$x(t) = D(t)x(T_k) + v(t), \quad t \in [T_k, T_{k+1}), \tag{9}$$

where $D(t)$ and $v(t)$ are time dependent.

Let $u \in \mathbb{R}^N$ an arbitrary vector. Define

$$f^k := \langle u, \frac{1}{T} \int_{T_k}^{T_{k+1}} x(t)\mathrm{d}t \rangle, \tag{10}$$

the benefit attached to the time period $[T_k, T_{k+1})$ after k times of re-assignment. Then the average benefit after K operations is given by

$$\frac{1}{K} \sum_{k=0}^{K} f^k.$$

According to (9) and by the definition of P, we have

$$x(T_{k+1}) = P(Dx(T_k) + v). \tag{11}$$

Lemma

Given $k \in \mathbb{N}$ and $P \in \mathcal{P}$, the matrix $\mathcal{I}_N - (PD)^k$ is invertible.

Theorem (One periodic [3])

$(x(T_k))_{k\in\mathbb{N}}$ is a constant sequence and we have for all $k \in \mathbb{N}$

$$x(T_k) = (\mathcal{I}_N - PD)^{-1}Pv.$$

The result shows that every $KT-$periodic evolution will actually be $T-$periodic.

Optimization problem

$$\max_{P \in \mathcal{P}} J(P) := \max_{P \in \mathcal{P}} \langle u, (\mathcal{I}_{N_z} - PD)^{-1} Pv \rangle, \tag{12}$$

Remark

Since $\#\mathfrak{S} = N!$, this problem cannot be tackled in realistic cases where large values of N must be considered, e.g., to keep a good numerical accuracy.

Expand the functional (12) as follows

$$\langle u, (\mathcal{I}_{N_z} - PD)^{-1} Pv \rangle = \sum_{l=0}^{+\infty} \langle u, (PD)^l Pv \rangle = \langle u, Pv \rangle + \sum_{l=1}^{+\infty} \langle u, (PD)^l Pv \rangle,$$

Approximation problem

$$\max_{P \in \mathcal{P}} J^{\text{approx}}(P) := \max_{P \in \mathcal{P}} \langle u, Pv \rangle. \tag{13}$$

Lemma

Let $\sigma_+, \ \sigma_- \in \mathfrak{S}$ *such that* $v_{\sigma_+(1)} \leq v_{\sigma_+(2)} \cdots \leq v_{\sigma_+(N)}$ *and* $v_{\sigma_-(N)} \leq v_{\sigma_-(N-1)} \leq \cdots \leq v_{\sigma_-(1)}$ *and* $P_+, \ P_- \in \mathcal{P}$*, the corresponding permutation matrices. Then*

$$P_+ = \mathrm{argmax}_{P \in \mathcal{P}} J^{approx}(P), \quad P_- = \mathrm{argmin}_{P \in \mathcal{P}} J^{approx}(P).$$

Remark (Optimal matrix)

- P_+: *associates the largest coefficient of u with the largest coefficient of v, the second largest coefficient with the second largest, and so on.*
- P_-: *associates the largest coefficient of u with the smallest coefficient of v, the second largest coefficient with the second smallest, and so on.*

Theorem (Criterion [3])

Assume that u and v have positive entries and define

$$\phi(m_1) := \frac{1}{s_{\lceil \frac{m_1}{2} \rceil}} \Big(\sum_{l=1}^{+\infty} d_{\max}^l F_{(l+1)m_1}^+ - d_{\min}^l F_{(l+1)m_1}^- \Big), \tag{14}$$

where $m_1 := \# \{n = 1, \ldots, N \mid \sigma(n) \neq \sigma_+(n)\}$, $d_{\max} := \max_{n=1,\ldots,N}(d_n)$ *and* $d_{\min} := \min_{n=1,\ldots,N}(d_n)$. *Assume that:*

$$\max_{m_1 \geq 2} \phi(m_1) \leq 1. \tag{15}$$

Then the problem $\max_{P\in\mathcal{P}}\langle u, (\mathcal{I}_{N_z} - PD)^{-1}Pv\rangle$ *(resp.* $\min_{P\in\mathcal{P}}\langle u, (\mathcal{I}_{N_z} - PD)^{-1}Pv\rangle$*) and the problem* $\max_{P\in\mathcal{P}}\langle u, Pv\rangle$ *(resp.* $\min_{P\in\mathcal{P}}\langle u, Pv\rangle$*) have the same solution.*

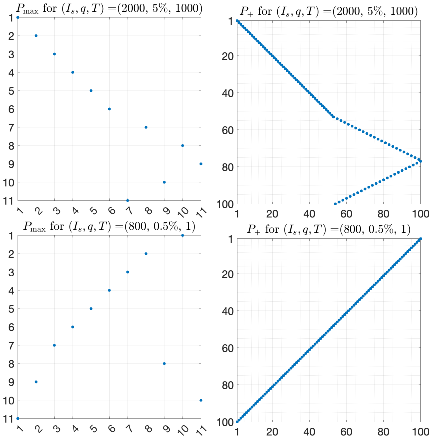


Figure: Optimal matrix $P_{\max}$ for Problem (12) and $N = 11$ (Left) and P_+ for Problem (13) and $N = 100$ (Right) for the two parameters triplets. The blue points represent non-zero entries, i.e., entries equal to 1.

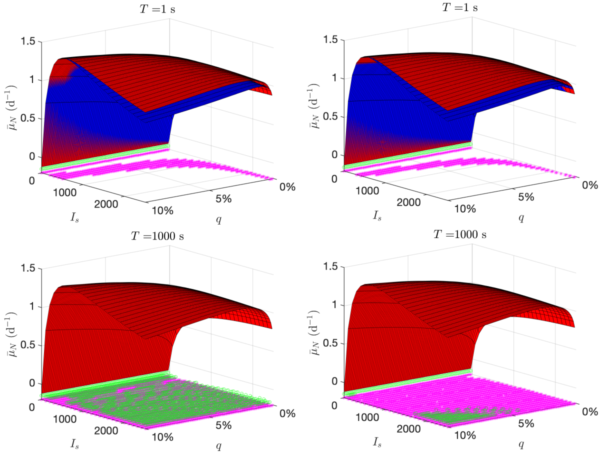


Figure: Average net specific growth rate $\bar{\mu}_N$ for $T = 1\,\mathrm{s}$ (Top) and for $T = 1000\,\mathrm{s}$ (Bottom). Left: $N = 5$. Right: $N = 9$. The red surface is obtained with $P_{\max}$ and the blue surface is obtained with P_+. The purple stars represent the cases where $P_{\max} = P_+$ or, in case of multiple solution, $\bar{\mu}_N(P_{\max}) = \bar{\mu}_N(P_+)$. The green circle represent the cases where the criterion (15) is satisfied.

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