Dirichlet-Neumann and Neumann-Neumann Methods for Elliptic Control Problems

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Abstract

We present the Dirichlet-Neumann (DN) and Neumann-Neumann (NN) methods applied to the optimal control problems arising from elliptic partial differential equations (PDEs) under the H^{-1} regularization. We use the Lagrange multiplier approach to derive a forward-backward optimality system with the L^2 regularization, and a singular perturbed Poisson equation with the H^{-1} regularization. The H^{-1} regularization thus avoids solving a coupled bi-Laplacian problem, yet the solutions are less regular. The singular perturbed Poisson equation is then solved by using the DN and NN methods, and a detailed analysis is given both in the one-dimensional and two-dimensional case. Finally, we provide some numerical experiments with conclusions.

1 Introduction

Consider the state $y(\mathbf{x})$ governed by the elliptic partial differential equation (PDE)

$$-\operatorname{div}(\kappa(\mathbf{x})\nabla y(\mathbf{x})) = u(\mathbf{x}), \quad \mathbf{x} \in \Omega, \qquad y(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$ is a bounded domain and $\partial\Omega$ its boundary. Here u is a control variable from an admissible set U_{ad}, which drives the state y to a target state $\hat{y}$. Problem (1) originates from the stationary heat conduction equation. In this setting, $\kappa(\mathbf{x})$ denotes the thermal conductivity of Ω, $y(\mathbf{x})$ is the temperature at a particular position $\mathbf{x}$ and $u(\mathbf{x})$ represents a controlled heat source. The goal is to find the optimal control variable u^* which minimizes the cost functional for $\nu \in \mathbb{R}^+$,

$$J(y, u) = \frac{1}{2}\int_\Omega |y(\mathbf{x}) - \hat{y}(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} + \frac{\nu}{2}\|u\|^2_{U_{\mathrm{ad}}}, \tag{2}$$

subject to the constraint (1). The term $\frac{\nu}{2}\|u\|^2_{U_{\mathrm{ad}}}$ can be considered as the cost of applying such a control u. It is said that the control is expensive if ν is large.

From a mathematical viewpoint, the presence of this term with $\nu \in \mathbb{R}^+$ has a regularizing effect on the optimal control.

The analysis of Domain Decomposition methods (DDMs) for the elliptic PDE (1) is well established, see for instance [12]. Much less is known for DD methods applied to PDE-constrained optimal control problems, see for instance [5, 6]. Although the admissible set U_{ad} is often considered as $L^2(\Omega)$ for such elliptic control problems, a recent study shows that the energy space $H^{-1}(\Omega)$ can also be used for the regularization [10]. Moreover, this space can be expanded with $L^2(0, T; H^{-1}(\Omega))$ to treat parabolic control problems [7]. From an analytical point of view, the first-order optimality system can be simplified to a Poisson type equation by using the energy space $H^{-1}(\Omega)$, whereas a biharmonic type problem still needs to be treated for the usual $L^2(\Omega)$ regularization. Moreover, applications of the energy norm can also be found in electrical engineering, fluid mechanics [9], etc.

Inspired by this approach, we study in this paper DDMs applied to the optimal control problem (1)-(2) using the energy norm. More precisely, we introduce in Section 2 the use of the energy norm H^{-1} for the elliptic control problem, and compare the optimality system with that of the L^2 norm. Although we consider for simplicity an unconstrained control, this can be extended to problems with state or control constraints, see also [13]. We then provide in Section 3 a convergence analysis of the Dirichlet-Neumann (DN) [1] and the Neumann-Neumann (NN) [2] methods applied to the optimality system. Some numerical experiments are given in Section 4, where we conclude with some comments.

2 Regularization: L^2 vs H^{-1}

We assume that both the control u and the target state $\hat{y}$ are in $L^2(\Omega)$, and consider first $U_{\text{ad}} = L^2(\Omega)$ as the set of all feasible controls. Using the Lagrange multiplier approach [13], we get for the first-order optimality system for problem (1)-(2)

$$\begin{aligned} -\text{div}\,(\kappa(\mathbf{x})\nabla y(\mathbf{x})) &= u(\mathbf{x}), \quad \mathbf{x} \in \Omega, & y(\mathbf{x}) &= 0, \quad \mathbf{x} \in \partial\Omega, \\ -\text{div}\,(\kappa(\mathbf{x})\nabla p(\mathbf{x})) &= y(\mathbf{x}) - \hat{y}(\mathbf{x}), \quad \mathbf{x} \in \Omega, & p(\mathbf{x}) &= 0, \quad \mathbf{x} \in \partial\Omega, \\ p(\mathbf{x}) + \nu u(\mathbf{x}) &= 0, \quad \mathbf{x} \in \Omega, \end{aligned} \tag{3}$$

where p is the Lagrange multiplier (or adjoint state). Inserting the third equation of (3) into the first equation, and the result into the second equation, we can rewrite the optimality system (3) with one single variable, for instance, with respect to the state variable y as

$$\begin{aligned} \nu\text{div}\Big(\kappa(\mathbf{x})\nabla\big(\text{div}\,(\kappa(\mathbf{x})\nabla y(\mathbf{x}))\big)\Big) + y(\mathbf{x}) &= \hat{y}(\mathbf{x}), & \mathbf{x} &\in \Omega, \\ \text{div}\,(\kappa(\mathbf{x})\nabla y(\mathbf{x})) = y(\mathbf{x}) &= 0, & \mathbf{x} &\in \partial\Omega. \end{aligned} \tag{4}$$

In particular, we identify in (4) a biharmonic operator by taking the conductivity $\kappa(\mathbf{x}) = 1$ everywhere over the domain.

We consider now $U_{\text{ad}} = H^{-1}(\Omega)$ in (2) as the set of all feasible controls. As proposed in [10], we can define the norm in $H^{-1}(\Omega)$ by

$$\|u\|^2_{H^{-1}(\Omega)} := \|\sqrt{\kappa}\nabla y\|^2_{L^2(\Omega)}, \tag{5}$$

which is the energy norm. Note that the conductivity κ is positive. On the other hand, following the same reasoning as in the $L^2(\Omega)$ case to derive the optimality system, we obtain

$$-\nu \text{div}\left(\kappa(\mathbf{x})\nabla y(\mathbf{x})\right) + y(\mathbf{x}) = \hat{y}(\mathbf{x}), \quad \mathbf{x} \in \Omega, \qquad y(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega. \tag{6}$$

Comparing (6) with the reduced optimality system under L^2 regularization (4), we observe that indeed only a Laplace type operator needs to be solved in (6).

Remark 1. *We need to be careful when comparing solutions of the two reduced optimality systems* (4) *and* (6)*, since we penalize the control in different norms and solve different equations. In the L^2 case, the control can be determined by $u = -\frac{1}{\nu}p$ which is proportional to the adjoint state variable, while it is proportional to the state variable in the H^{-1} case, since $u = \frac{1}{\nu}(\hat{y} - y)$. Furthermore, the solution is less regular in the H^{-1} case as shown in [10].*

Remark 2. *Depending on the value of ν,* (6) *is a singularly perturbed PDE. Standard numerical methods can perform poorly, we refer to the monograph [11] for a review of robust numerical methods for such problems. In the recent work [8], the authors use an algebraic multigrid method and a balancing domain decomposition by constraints preconditioner for a finite element discretization to treat the problem* (6)*. They observed that optimal convergence is ensured with $\nu = h^2$, h being the mesh size.*

3 Convergence Analysis of DD methods

We now provide a convergence analysis for the DN and the NN methods applied to solve the reduced optimality system (6), and then compare with DN and NN methods applied to (4) from [5].

Without loss of generality, the analysis is given under the assumption that the target state $\hat{y} = 0$, meaning that we focus on the error equation related to (6). Moreover, we assume that the conductivity coefficient $\kappa(x) = 1$ everywhere over the domain for the following analysis, although the DN and NN methods are defined for a general $\kappa(x)$. Let us first consider the one-dimensional case with the domain $\Omega = (0, 1)$. We decompose it into two non-overlapping subdomains $\Omega_1 = (0, \alpha)$ and $\Omega_2 = (\alpha, 1)$ with α the interface. We denote by e_i the error in domain Ω_i for $i = 1, 2$.

For the DN method, the error equations for (6) are for iteration index $n = 1, 2, \ldots$,

$$\begin{aligned} &\partial_{xx}e_1^n - \nu^{-1}e_1^n = 0, \quad e_1^n(0) = 0, \quad e_1^n(\alpha) = e_\alpha^{n-1}, \\ &\partial_{xx}e_2^n - \nu^{-1}e_2^n = 0, \quad e_2^n(1) = 0, \quad \partial_x e_2^n(\alpha) = \partial_x e_1^n(\alpha), \end{aligned} \tag{7}$$

with $e_\alpha^n := (1-\theta)e_\alpha^{n-1} + \theta e_2^n(\alpha)$ and $\theta \in (0,1)$ a relaxation parameter. We notice that the error equations (7) are similar to the ones in [4, Equation (2.4)] for applying the Dirichlet-Neumann waveform relaxation (DNWR) method to the heat equation. Indeed, after a Laplace transform, the error equations for the DNWR method in the one dimensional case are like (7), where ν^{-1} is replaced by s. For this reason, we follow the same calculations as in [4] and find the convergence factor

$$\rho_{\mathrm{DN}} := \left|1 - \theta\left[1 + \tanh\left(\sqrt{\nu^{-1}}(1-\alpha)\right)\coth\left(\sqrt{\nu^{-1}}\alpha\right)\right]\right|. \tag{8}$$

This leads us to the following convergence results.

Theorem 1. *The DN method with $\theta = 1$ applied to Problem (6) converges if and only if the interface is closer to the right boundary (i.e., $\alpha > \frac{1}{2}$).*

Proof. Taking $\theta = 1$ in (8), we obtain the convergence factor

$$\rho_{\mathrm{DN}} = \tanh\left(\sqrt{\nu^{-1}}(1-\alpha)\right)\coth\left(\sqrt{\nu^{-1}}\alpha\right),$$

that is smaller than 1 if and only if $\alpha > \frac{1}{2}$ which can be seen by studying the function $f(x) = \sinh(1-x)\cosh(x) - \cosh(1-x)\sinh(x)$ for $x \in [0,1]$. □

Theorem 2. *For symmetric subdomains (i.e., $\alpha = \frac{1}{2}$), the convergence of the DN method for Problem (6) is linear and is independent of the value of the regularization parameter ν. It converges in two iterations if $\theta = \frac{1}{2}$.*

Proof. We just have to take $\alpha = \frac{1}{2}$ in (8) and finds $\rho_{\mathrm{DN}} = |1-2\theta|$. □

Theorem 3. *For asymmetric subdomains (i.e., $\alpha \neq \frac{1}{2}$), the DN method converges for Problem (6) if and only if*

$$0 < \theta < 2\theta_{DN}^\star, \quad \theta_{DN}^\star := \frac{1}{1 + \tanh\left(\sqrt{\nu^{-1}}(1-\alpha)\right)\coth\left(\sqrt{\nu^{-1}}\alpha\right)}. \tag{9}$$

Moreover, it converges in two iterations if and only if $\theta = \theta_{DN}^\star$.

Proof. From the convergence factor (8), the interior part of the absolute value is smaller than 1, since $\theta \in (0,1)$ and $1 + \tanh\left(\sqrt{\nu^{-1}}(1-\alpha)\right)\coth\left(\sqrt{\nu^{-1}}\alpha\right)$ is strictly positive. We then just need to ensure that

$$\theta\left[1 + \tanh\left(\sqrt{\nu^{-1}}(1-\alpha)\right)\coth\left(\sqrt{\nu^{-1}}\alpha\right)\right] < 2,$$

which leads to the inequality in (9). On the other hand, we find directly $\theta_{\mathrm{DN}}^\star$ by equating (8) to zero. □

Remark 3. *As expected, we find similar results in the symmetric case as for the L^2 regularization. However, we have an optimal relaxation parameter for asymmetric decompositions, which is strictly smaller than 1, whereas a pair of parameters is needed for the L^2 regularization which can be greater than one in some cases, see [5]. This is due to the fact that two transmission conditions need to be considered for a biharmonic type problem.*

The error equations for the NN method, for iteration index $n = 1, 2, \cdots$, are

$$\partial_{xx} e_j^n - \nu^{-1} e_j^n = 0, \quad e_1^n(0) = 0, \quad e_2^n(1) = 0, \quad e_j^n(\alpha) = e_\alpha^{n-1}, \tag{10}$$

where the transmission condition is given by $e_\alpha^n := e_\alpha^{n-1} - \theta\big(\psi_1^n(\alpha) + \psi_2^n(\alpha)\big)$ and ψ_j^n satisfies the correction step

$$\partial_{xx} \psi_j^n - \nu^{-1} \psi_j^n = 0, \quad \psi_1^n(0) = 0, \quad \psi_2^n(1) = 0, \quad \partial_{n_j} \psi_j^n(\alpha) = \partial_{n_1} e_1^n(\alpha) + \partial_{n_2} e_2^n(\alpha). \tag{11}$$

Solving (10)-(11) on each domain Ω_j and applying the boundary conditions at $x = 0$ and $x = 1$, we find the solutions with A^n, B^n, C^n, D^n four coefficients to be determined for e_1^n, e_2^n, ψ_1^n and ψ_2^n. Evaluating then e_j^n at $x = \alpha$, and using the transmission condition $e_j^n(\alpha) = e_\alpha^{n-1}$, we can determine the two coefficients A^n, B^n and get

$$e_1^n(x) = e_\alpha^{n-1} \frac{\sinh(\sqrt{\nu^{-1}} x)}{\sinh(\sqrt{\nu^{-1}} \alpha)}, \quad e_2^n(x) = e_\alpha^{n-1} \frac{\sinh\Big(\sqrt{\nu^{-1}}(1-x)\Big)}{\sinh\Big(\sqrt{\nu^{-1}}(1-\alpha)\Big)}. \tag{12}$$

Similarly, we evaluate $\partial_{n_j} \psi_j^n$ at $x = \alpha$, and using the transmission condition $\partial_{n_j} \psi_j^n(\alpha) = \partial_{n_1} e_1^n(\alpha) + \partial_{n_2} e_2^n(\alpha)$ with the help of (12), we can determine the remaining two coefficients C^n, D^n and get,

$$\psi_1^n(x) = e_\alpha^{n-1} \frac{\sinh(\sqrt{\nu^{-1}} x)}{\cosh(\sqrt{\nu^{-1}} \alpha)} \Big(\coth(\sqrt{\nu^{-1}} \alpha) + \coth(\sqrt{\nu^{-1}}(1-\alpha))\Big),$$

$$\psi_2^n(x) = e_\alpha^{n-1} \frac{\sinh\Big(\sqrt{\nu^{-1}}(1-x)\Big)}{\cosh(\sqrt{\nu^{-1}}(1-\alpha))} \Big(\coth(\sqrt{\nu^{-1}} \alpha) + \coth(\sqrt{\nu^{-1}}(1-\alpha))\Big).$$

Using finally the definition of the transmission condition e_α^n, we find the convergence factor

$$\begin{aligned} \rho_{\mathrm{NN}} := \Big|1 - \theta\Big(&\tanh(\sqrt{\nu^{-1}}\alpha) + \tanh\Big(\sqrt{\nu^{-1}}(1-\alpha)\Big)\Big) \\ &\times \Big(\coth(\sqrt{\nu^{-1}}\alpha) + \coth(\sqrt{\nu^{-1}}(1-\alpha))\Big)\Big|. \end{aligned} \tag{13}$$

We obtain the following convergence results.

Theorem 4. *For symmetric subdomains (i.e., $\alpha = \frac{1}{2}$), the convergence of the NN method for Problem (6) is linear and is independent of the value of the regularization parameter ν. It converges in two iterations if $\theta = \frac{1}{4}$.*

Proof. We just have to take $\alpha = \frac{1}{2}$ in (13) and find $\rho_{\mathrm{NN}} = |1 - 4\theta|$. □

Theorem 5. *For asymmetric subdomains (i.e., $\alpha \neq \frac{1}{2}$), the NN method converges for Problem (6) if and only if*

$$0 < \theta < 2\theta_{NN}^\star, \quad \theta_{NN}^\star := \frac{1}{\Big(\tanh(\sqrt{\nu^{-1}}\alpha) + \tanh\big(\sqrt{\nu^{-1}}(1-\alpha)\big)\Big)\Big(\coth(\sqrt{\nu^{-1}}\alpha) + \coth(\sqrt{\nu^{-1}}(1-\alpha))\Big)}. \tag{14}$$

Furthermore, it converges in two iterations if and only if $\theta = \theta_{NN}^\star$.

Proof. Following the same steps as in the proof of Theorem 3, we obtain the inequality (14), and we find directly $\theta^\star_{\mathrm{NN}}$ by equating (13) to zero. $\square$

Remark 4. *As shown in Theorem 3 and in Theorem 5, both the DN and the NN methods converge in two iterations to the exact solution. Moreover, we have a bound for the relaxation parameter θ of each method for which the convergence of the method is guaranteed.*

The above analysis can also be extended to the two-dimensional case. More precisely, we assume that the domain Ω is now given by $[0,1]\times[0,1]$, which is then divided into two non-overlapping subdomains $\Omega_1 = (0,\alpha)\times[0,1]$ and $\Omega_2 = (\alpha,1)\times[0,1]$, with the interface at $x_1 = \alpha$ denoted by $\Gamma := \{\alpha\}\times[0,1]$. In addition, we keep the assumption that $\hat{y} = 0$ and $\kappa(x) = 1$. The two-dimensional analysis is often carried out by using a Fourier expansion in one direction, in our case, the x_2 direction $e_i^n(x_1,x_2) = \sum_{k=0}^{\infty} \hat{e}_i(x_1,k)\sin(k\pi x_2)$. In this way, the error function related to $e_i(x_1,x_2)$ passes to $\hat{e}_i(x_1,k)$, and for instance, in the DN case is governed by

$$
\begin{aligned}
&\partial_{x_1x_1}\hat{e}_1^n - \frac{\nu k^2\pi^2+1}{\nu}\hat{e}_1^n = 0, \quad \hat{e}_1^n(0,k) = 0, \quad \hat{e}_1^n(\alpha,k) = \hat{e}_\alpha^{n-1},\\
&\partial_{x_1x_1}\hat{e}_2^n - \frac{\nu k^2\pi^2+1}{\nu}\hat{e}_2^n = 0, \quad \hat{e}_2^n(1,k) = 0, \quad \partial_{x_1}\hat{e}_2^n(\alpha,k) = \partial_{x_1}\hat{e}_1^n(\alpha,k),
\end{aligned}
\tag{15}
$$

with $\hat{e}_\alpha^n := (1-\theta)\hat{e}_\alpha^{n-1} + \theta\hat{e}_2^n(\alpha,k)$ and $\theta\in(0,1)$. We observe that (15) has the same structure as in the one-dimensional case (7), where ν^{-1} is replaced by $\frac{\nu k^2\pi^2+1}{\nu}$. Therefore, the same type of reasoning can be applied to analyze this iteration, and we have the following results.

Theorem 6. *For symmetric subdomains (i.e., $\alpha = \frac{1}{2}$), the convergence of the DN and the NN methods for Problem* (6) *are both linear and independent of the value of ν. It converges in two iterations if $\theta = \frac{1}{2}$ for the DN method and $\theta = \frac{1}{4}$ for the NN method.*

Theorem 7. *For asymmetric subdomains (i.e., $\alpha \neq \frac{1}{2}$), the DN method converges for Problem* (6) *whenever*

$$
\rho_{DN2d} := \sup_{k\in\mathbb{N}}\left|1-\theta\left[1+\tanh\left(\sqrt{\frac{\nu k^2\pi^2+1}{\nu}}(1-\alpha)\right)\coth\left(\sqrt{\frac{\nu k^2\pi^2+1}{\nu}}\alpha\right)\right]\right| < 1.
\tag{16}
$$

The NN method converges for Problem (6) *whenever*

$$
\begin{aligned}
\rho_{NN2d} := \sup_{k\in\mathbb{N}}\Big|1-\theta\Big(&\tanh\Big(\sqrt{\frac{\nu k^2\pi^2+1}{\nu}}\alpha\Big)+\tanh\left(\sqrt{\frac{\nu k^2\pi^2+1}{\nu}}(1-\alpha)\right)\Big)\\
&\cdot\Big(\coth\Big(\sqrt{\frac{\nu k^2\pi^2+1}{\nu}}\alpha\Big)+\coth\Big(\sqrt{\frac{\nu k^2\pi^2+1}{\nu}}(1-\alpha)\Big)\Big)\Big| < 1.
\end{aligned}
\tag{17}
$$

4 Numerical experiments

In this section, we provide numerical experiments to illustrate the convergence rate of the DN and the NN methods for Problem (1)-(2) with $\nu = 1$ and $\hat{y} = 0$. Figure 1 (top) shows the one-dimensional convergence behaviour of these two methods for different choices of θ with an asymmetric decomposition $\alpha = \frac{1}{3}$. The best choices of the relaxation parameter are given by $\theta^\star_{\mathrm{DN}} \approx 0.355$ and $\theta^\star_{\mathrm{NN}} \approx 0.229$. In particular, we observe some divergence behavior in the case of the NN method for $\theta = 0.5$ and $\theta = 0.7$. Indeed, this corresponds to the result in Theorem 5, since these two values are greater than $2\theta^\star_{\mathrm{NN}}$ which is the upper bound for the relaxation parameter θ. Furthermore, we observe the convergence to the exact solution in two iterations for a non-symmetric domain decomposition, whereas a three-step convergence is needed for the L^2 regularization [5]. Figure 1 (bottom) presents the behavior of the convergence factors (16) and (17) in the two-dimensional case. The interface here is chosen to be asymmetric $\Gamma = \{\frac{1}{3}\} \times [0, 1]$. We observe good convergence behaviors for some tested relaxation parameters θ. Furthermore, the NN method does not converge for $\theta = 0.5$ and $\theta = 0.7$ as in the one-dimensional case. We obtain that $\rho_{\mathrm{DN2d}} \approx 0.173$ for $\theta^\star_{\mathrm{DN2d}} \approx 0.414$ and $\rho_{\mathrm{NN2d}} \approx 0.046$ for $\theta^\star_{\mathrm{NN2d}} \approx 0.239$. These two optimal relaxation parameters can also be found by equioscillating the value of the convergence factor both at $k = 0$ and $k \to \infty$. Moreover for each method, we find that these optimal relaxation parameters stay very close between the one-dimensional and the two-dimensional case.

To conclude, we presented a convergence analysis of the DN and the NN methods for elliptic optimal control problems using the energy norm for regularization. Only one Poisson type equation needs to be solved, whereas a biharmonic type equation is required for L^2 regularization. Under the energy norm, we found similar results in the symmetric case as for the Poisson problem. Therefore, we can expect similar convergence behavior for many subdomains as presented in [3]. Furthermore, explicit formulations along with an upper bound are also given for the optimal relaxation parameters with a non-symmetric decomposition, for which the methods converge still in two iterations in the one-dimensional case.

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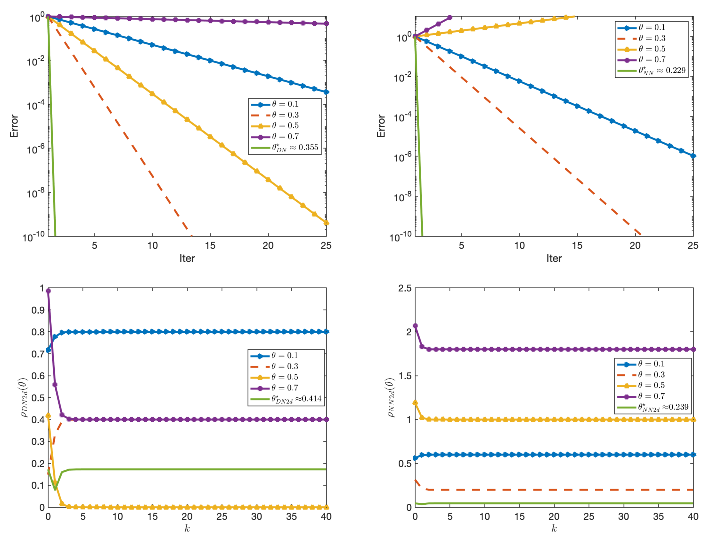

Figure 1: Error decay in 1D w.r.t. the number of iterations for the DN method (top-left) and the NN method (top-right) with the interface at $\alpha = \frac{1}{3}$. Convergence factors (16) and (17) in 2D w.r.t. the value of $k \in [0, 40]$ for the DN method (bottom-left) and the NN method (bottom-right) with the interface at $\Gamma = \{\frac{1}{3}\} \times [0, 1]$.

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